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The micromechanics theory of classical plates: a congruous estimate of overall elastic stiffness

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Abstract

A micromechanics theory is set forth for classical, or Love–Kirchhoff plate. A generalized eigenstrain theory, an eigen-curvature formulation, is proposed, which can be viewed as the analogue, or counterpart of the eigenstrain formulation in linear elasticity. This thin plate version of micromechanics is capable of dealing with heterogeneous inclusions, or inhomogeneities, whose size is comparable with the thickness of thin plates, under these circumstances, the continuum micromechanics theory is no longer applicable. The paper consists of three parts. In the first part, the solution of the elliptical inclusion in an infinite thin plate is revised. In the second part, several variational inequalities of the Love–Kirchhoff plate are derived, including a Hashin–Shtrikman/Talbot–Willis type principle. In the third part, as an application, exact variational estimates are given to bound the effective elastic stiffness, and a self-consistent scheme is also discussed. The newly derived bounds are congruous with Love–Kirchhoff plate theory, i.e. they are genetic to the governing equations of Love–Kirchhoff plate. They may serve as the alternatives together with the Hashin–Shtrikman bounds in linear elastostatics in the design process of composite plates.

Keywords: Micromechanics; Love–Kirchhoff plate theory; Eigen-curvatuare; Variational bound; Effective elastic stiffness

1. Introduction

The micromechanics of elastic and inelastic materials have been expanded vastly in the past two decades (Mura, 1987; Nemat-Nasser and Hori, 1993 and references therein). The main impetus for such progress is the need for understanding behaviors of various composite materials, which occupy the center stage of modern technology. However, from civil engineering, or structural mechanics point of view, there is still a gap between the established micromechanics theory and the engineering design practice. When structures are made of composite materials (indeed they are, most of them, if not all of them), to assess the overall elastic stiffness of a structure, one has to consider the particular type of structure theory that is chosen in the design process. In practice, when a composite plate has a definitive

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structure, such as a sandwich plate, or laminar plate in general, to evaluate the overall elastic stiffness of the plate structure, a class of composite plate theories have been developed in last half century, which take into account the specific micro-structure of the plate (e.g. Christensen, 1979; Parton and Kudryavtsev, 1993). Nevertheless, for the composite structures that are made of several random-distributed heterogeneous materials, to this author's knowledge, there is no systematic theory available to evaluate the overall effective elastic stiffness of the structure.

The folk wisdom on the subject is: if a structure is made of random-distributed, or isotropic unstructured, composite materials, one may first evaluate the overall effective elastic modulus by taking the object as a three-dimensional (3D) elastic medium, and then apply the particular structural theory to the equivalent elastic body. There are flaws or limitations in this type of design philosophy. In general, the kinematic assumptions of different structural theories, the director theory of plates and shells for instance (Naghdi, 1963, 1972), always introduce internal constrains on constitutive relations, such that, the approximated constitutive relations, or the structural constitutive relations are always anisotropic in character, and consequently differ from the constitutive relations of 3D isotropic elastic body, because the structural constitutive relations are special "projection" of 3D constitutive relations. For thin plate, in particular, the Love–Kirchhoff hypothesis (Love, 1926) assumes that the transverse shear modulus within the plate's cross section is infinitive, and consequently the transverse shear deformation of the cross section is always zero under the finite shear resultance, which is the source of induced "anisotropy". Lucid accounts on projection of linear elastic constitutive laws onto plates and shells are given by Naghdi (1963, 1972). Apparently, there are fundamental distinctions between the concept of elastic modulus in a continuum and the concept of elastic stiffness in a specific structure, though, unfortunately, these differences are sometimes overlooked by practitioners. It is our opinion that any unscrupulous use of the overall elastic modulus derived from micromechanics of linear elasticity will not be consistent with the type of structural theory adopted in design process.

In addition, continuum micromechanics is based on the notion of *representative volume element* (Hill, 1963a). If the size of an inclusion is large enough in comparison with the thickness of a plate (mathematically speaking, if they are in the same order), the concept of representative volume element will be no longer valid, because it has lost its original statistical implication. In reality, most structural components, such as beams, plates, and shells, are lower order dimensional manifolds, meaning that the characteristic length in thickness direction is much smaller than those in the other directions; thus, situations may occur that the size of inhomogeneities in a structure is in the same order of the thickness of the structure, which may lead to another type inaccuracies if one applies continuum micromechanics without discretion.

In this paper, we develop a micromechanics model within the framework of thin plate theory itself. The proposed congruous theory of micromechanics is based upon the following assumptions:

(i) The composite plate is assumed to fulfil all the kinematic assumptions of Love–Kirchhoff plates.
(ii) The composite plate is made of several different constituents, and each phase of the composite is dominant in the thickness direction of the plate, or at least approximately.

In other words, the composite plate that we are concerned with, is homogeneous in thickness direction of the plate, whereas its elastic stiffness, or compliance varies on the plates' middle surface. This allows us to replace the notion of representative volume in three-dimensional continuum by a notion of *representative surface area* on the plate's middle surfaces. Fig. 1 illustrates an ideal physical model of such composite thin plate, which is considered to be macroscopically isotropic within the plane surface. Since the composite plate is homogeneous in thickness direction of the plate, one may slice off its middle plane from the plate to obtain the representative surface area. In continuum microelasticity, the length scale of the representative element is unspecified; it can be viewed as a very small quantity macroscopically, whereas it can be also viewed as infinity microscopically. In the theory of Love–

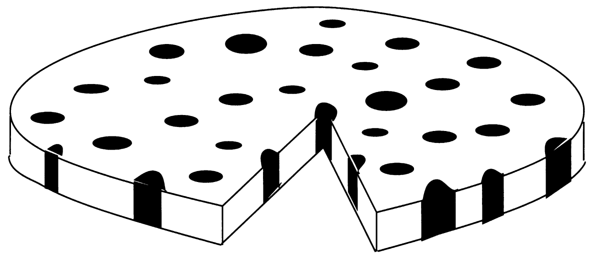


Fig. 1. The physical model of a thin plate made by unstructured composite materials.

Kirchhoff plate, the lateral dimension of the representative area element is bounded below; which has to be large enough comparing with the thickness of the plate, such that any boundary-value problem of the thin plate equations involved with a surface representative area element will make sense under the Love–Kirchhoff kinematic assumptions. In consequence, the Love–Kirchhoff plate theory introduces an intrinsic length scale h, which defines the scale for its associated "micromechanics". Nevertheless, the lateral dimensions of inhomogeneities, and inclusions within a representative surface area element are allowed to be arbitrarily small, since the governing equations of the thin plate is valid pointwise, or can be treated as "field equations". Another substantial restriction in this model is the requirement that a composite plate has to be homogeneous in thickness direction. In actual practice, such restriction on the distribution of inhomogeneities of the plate might be relaxed to as being approximately fulfilled.

In fact, the idea of developing independent micromechanics for structural theories is not new; Qin et al. (1991) apparently are the first group of people who realized that there is a parallel analogue between continuum micromechanics and structural micromechanics, and attempted to explore the possibilities. Unfortunately, the early study was reluctant to adopt the notion of eigen-curvature, showing sentimental string attached to continuum micromechanics, and it had not been carried out to final stage. The paper is organized as follows. In Section 2 and Appendices A and B, the inclusion theory of the thin plate is reformulated to amend the previous results. In passing, several elementary lemmas are stated, which are instrumental to establish the average ensemble properties of a composite thin plate. In Section 3, another important technical ingredient of micromechanics, the well-known Hashin–Shtrikman variational principle, is studied, whose presence is speculated to be in every conservative, elliptic systems. The recent extensions to piezoelectric medium and strain gradient dependent elasticity are clear evidences (Bisegna and Luciano, 1996; Smyshlyaev and Fleck, 1994). In Section 4, explicit bounds for thin plate's rigidity are discussed and a self-consistent estimate is also presented. The emphasis here is placed on the differences between the micromechanics of linear elasticity and that of Love–Kirchhoff plate theory.

2. Generalized eigenstrain (eigen-curvature) formulation

2.1. Preliminary

There are many treatises on thin plate theory; for our purpose, Love (1926) or Timoshenko and Woinowsky-Krieger (1959) is sufficient. For easy reference, the basic properties of thin plates are

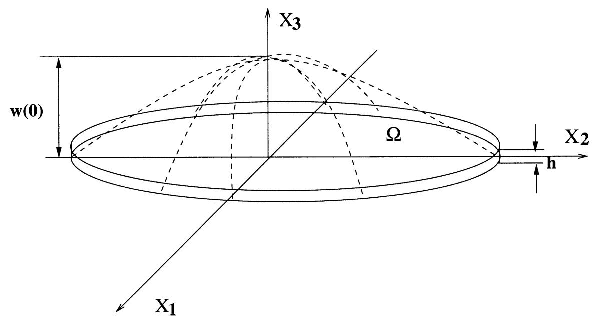


Fig. 2. The configuration of a thin plate.

provided when they are needed. Assume that Ω is a simply connected, bounded region in $\mathbb{R}^2$, which has a smooth boundary; subsequently, the normal vector, **n**, and tangential vector, **s**, on $\partial\Omega$ are uniquely defined[1]. The material space of the plate is defined as: $\Omega \times [-h/2, h/2] \subset \mathbb{R}^3$, (see Fig. 2), where h is the thickness of the plate made of composite material. We assume that the plate has n different phases, and each phase has a different elastic modulus. The deflection of the plate is defined as the mapping $w{:}\Omega \longmapsto \mathbb{R}$. The deformed middle surface of the plate is the graph $\{(\Omega, w) \subset \mathbb{R}^3\}$. As an a priori condition, all the kinematic assumptions (the Love–Kirchhoff for instance) about the plate are assumed automatically fulfilled for the composite plate under consideration.

The curvature of the plate is defined as:

$$k_{\alpha\beta} = -\frac{\partial^2 w}{\partial x_\alpha \partial x_\beta} \tag{1}$$

where (and in the rest of the paper) the Greek index always ranges from 1 to 2. The constitutive relations of linear elastic plate are expressed in terms of moments and curvatures:

$$m_{\alpha\beta} = L_{\alpha\beta\zeta\eta} k_{\zeta\eta} \tag{2}$$

$$k_{\alpha\beta} = N_{\alpha\beta\zeta\eta} m_{\zeta\eta} \tag{3}$$

where $L_{\alpha\beta\zeta\eta}$, and $N_{\alpha\beta\zeta\eta}$ are elastic stiffness and compliance tensors, respectively. The equilibrium equation of thin plates is

$$m_{\alpha\beta,\,\alpha\beta} + q = 0 \tag{4}$$

Let $M_n{:=}M_{\alpha\beta}n_\alpha n_\beta$, $M_{ns}{:=}M_{\alpha\beta}n_\alpha s_\beta$, $Q_n{:=}M_{\alpha\beta,\,\beta}n_\alpha$, where n_α, s_β are components of the normal and tangential vectors of $\partial\Omega$. The typical boundary conditions can be posed as

Clamped boundary conditions:

$$w = \hat{w}; \qquad \frac{\partial w}{\partial n} = \hat{\psi}_n \quad \forall \mathbf{x} \in \partial\Omega \tag{5}$$

[1] Consequently, we save some troubles in dealing with the corner conditions.

Force prescribed boundary conditions:

$$M_n = \hat{M}_n; \qquad V_n := Q_n + \frac{\partial M_{ns}}{\partial s} = \hat{V}_n \quad \forall \mathbf{x} \in \partial\Omega \tag{6}$$

Simply supported boundary condition:

$$w = \hat{w}; \qquad M_n = \hat{M}_n \quad \forall \mathbf{x} \in \partial\Omega \tag{7}$$

Mixed type boundary conditions can be posed by blending them together. In this paper, we are mainly interested in the purely deflection prescribed boundary condition (5) and purely force prescribed boundary condition (6). According to convention, the deflection, w, that satisfies deflection prescribed boundary condition (5) is referred to as the kinematic admissible deflection field; whereas, the moment tensor, $m_{\alpha\beta}$, that satisfies Eq. (6) and the equilibrium equation (4) is referred to as the statically admissible moment field. Before proceeding further, it would be expedient to review some basic formulas of thin plates.

Lemma 2.1. *Let* $W_0(\Omega) = \{w | w \in H^1(\Omega),\ w = \frac{\partial w}{\partial n} = 0\}$, $k_{\alpha\beta} = -w_{,\alpha\beta}$ and $m^0_{\alpha\beta} \in c^2(\Omega)$. *The condition*

$$\iint_\Omega m^0_{\alpha\beta} k_{\alpha\beta}\, \mathrm{d}\Omega = 0, \quad \forall w \in W_0(\Omega) \tag{8}$$

implies that: $m^0_{\alpha\beta,\,\alpha\beta} = 0$, $\forall \mathbf{x} \in \Omega$.

Proof. Integration by parts yields

$$\iint_\Omega m^0_{\alpha\beta} k_{\alpha\beta}\, \mathrm{d}\Omega = \oint_{\partial\Omega} \left(Q^0_n + \frac{\partial M^0_{ns}}{\partial s} \right) w\, \mathrm{d}S - \oint_{\partial\Omega} m^0_n \frac{\partial w}{\partial n}\, \mathrm{d}S - \iint_\Omega m^0_{\alpha\beta,\,\alpha\beta} w\, \mathrm{d}\Omega = -\iint_\Omega m^0_{\alpha\beta,\,\alpha\beta} w\, \mathrm{d}\Omega$$

The claim then follows immediately. ♣

Lemma 2.2. *Let* $k_{\alpha\beta} \in C^2(\Omega)$, $m_{\alpha\beta} \in C^2_0(\Omega)$, *and* $m_{\alpha\beta,\,\alpha\beta} = 0$. *The condition*

$$\iint_\Omega m_{\alpha\beta} k_{\alpha\beta}\, \mathrm{d}\Omega = 0 \tag{9}$$

implies

$$\epsilon_{\alpha\gamma} k_{\alpha\beta,\,\beta\gamma} = 0, \quad \forall \mathbf{x} \in \mathrm{int}\{\Omega\} \tag{10}$$

if $k_{\alpha,\,\beta}$ *are smooth enough.*

Note that where $\epsilon_{\alpha\beta}$, is the 2D permutation symbol, which is defined as

$$\epsilon_{\alpha\beta} = \begin{cases} 1; & \alpha > \beta \\ 0; & \alpha = \beta \\ -1; & \alpha < \beta \end{cases} \tag{11}$$

Proof. Since $m_{\alpha\beta,\,\alpha\beta} = 0$, there exists smooth function $\phi \in C^4_0(\Omega)$ such that

$$m_{\alpha\beta} = \epsilon_{\alpha\gamma}\phi_{,\gamma\beta} \tag{12}$$

By Gauss theorem and integration by parts,

$$\int\int_{\Omega} k_{\alpha\beta}\epsilon_{\alpha\gamma}\phi_{,\gamma\beta}\,\mathrm{d}\Omega = \oint_{\partial\Omega}\epsilon_{\alpha\gamma}\phi_{,\gamma}k_{\alpha\beta}n_{\beta}\,\mathrm{d}S - \oint_{\partial\Omega}\epsilon_{\alpha\gamma}\phi k_{\alpha\beta,\,\beta}n_{\gamma}\,\mathrm{d}S + \int\int_{\Omega}\epsilon_{\alpha\gamma}k_{\alpha\beta,\,\beta\gamma}\phi\,\mathrm{d}\Omega = \int\int_{\Omega}\epsilon_{\alpha\gamma}k_{\alpha\beta,\,\beta\gamma}\phi\,\mathrm{d}\Omega = 0$$

Eq. (10) follows immediately. ♣

Lemma 2.3. *Suppose* $m_{\alpha\beta,\,\alpha\beta} = 0$. *The following identity holds*

$$\frac{1}{|\Omega|}\int\int_{\Omega} m_{\alpha\beta}\,\mathrm{d}\Omega = \frac{1}{2|\Omega|}\int_{\partial\Omega}\left\{m_{\zeta\alpha}x_{\beta}n_{\zeta} + m_{\eta\beta}x_{\alpha}n_{\eta} - m_{\zeta\eta,\,\eta}x_{\alpha}x_{\beta}n_{\zeta}\right\}\mathrm{d}S \tag{13}$$

Proof. First one may verify the following identity

$$m_{\alpha\beta} = \frac{\partial(m_{\zeta\alpha}x_{\beta})}{\partial x_{\zeta}} + \frac{\partial(m_{\eta\beta}x_{\alpha})}{\partial x_{\eta}} + \frac{1}{2}m_{\zeta\eta,\,\zeta\eta}x_{\alpha}x_{\beta} - \frac{1}{2}\frac{\partial^2(m_{\zeta\eta}x_{\alpha}x_{\beta})}{\partial x_{\zeta}\partial x_{\eta}} \tag{14}$$

If $m_{\zeta\eta,\,\zeta\eta} = 0$, then

$$m_{\alpha\beta} = \frac{\partial(m_{\zeta\alpha}x_{\beta})}{\partial x_{\zeta}} + \frac{\partial(m_{\eta\beta}x_{\alpha})}{\partial x_{\eta}} + \frac{1}{2}\frac{\partial^2(m_{\zeta\eta}x_{\alpha}x_{\beta})}{\partial x_{\zeta}\partial x_{\eta}} \tag{15}$$

Eq. (13) follows immediately by Gauss theorem. ♣
Define

$$\bar{f} := \langle f\rangle := \frac{1}{|\Omega|}\int\int_{\Omega} f\,\mathrm{d}\Omega. \tag{16}$$

It is relatively straightforward to show that

Lemma 2.4. *Suppose* $m_{\alpha\beta,\,\alpha\beta} = 0$, *the following identities hold*:

1.

$$\frac{1}{|\Omega|}\oint_{\partial\Omega}\big(m_{\alpha\beta}n_{\alpha} - \langle m_{\alpha\beta}\rangle n_{\alpha}\big)\big(w_{,\beta} - \langle w_{,\beta}\rangle\big)\,\mathrm{d}S + \frac{1}{|\Omega|}\oint_{\partial\Omega}\big(m_{\alpha\beta,\,\alpha}n_{\beta} - \langle m_{\alpha\beta,\,\alpha}\rangle n_{\beta}\big)\big(w - \langle w_{,\zeta}\rangle x_{\zeta}\big)\,\mathrm{d}S = \langle\mathbf{m}\rangle{:}\langle\mathbf{k}\rangle - \langle\mathbf{m}{:}\mathbf{k}\rangle \tag{17}$$

2.

$$\frac{1}{|\Omega|}\oint_{\partial\Omega}\left(m_{\alpha\beta}n_\alpha - \langle m_{\alpha\beta}\rangle n_\alpha\right)\left(w_{,\beta} + \langle k_{\gamma\beta}\rangle x_\gamma\right)\,\mathrm{d}S + \frac{1}{|\Omega|}$$

$$\oint_{\partial\Omega}\left(m_{\alpha\beta,\alpha}\,n_\beta - m^0_{\alpha\beta,\,\alpha}n_\beta\right)\left(w + \frac{1}{2}\langle k_{\zeta\eta}\rangle x_\zeta x_\eta\right)\,\mathrm{d}S = \frac{m^0_{\alpha\beta,\alpha}\,n_\beta}{|\Omega|} \tag{18}$$

$$\oint_{\partial\Omega}\left(w + \frac{1}{2}\langle k_{\zeta\eta}\rangle x_\zeta x_\eta\right)\,\mathrm{d}S + \langle\mathbf{m}\rangle{:}\langle\mathbf{k}\rangle - \langle\mathbf{m}{:}\mathbf{k}\rangle$$

where $m^0_{\alpha\beta,\,\beta}$ *is any constant.* ♣

2.2. The solution of elliptical inclusion problem

The elliptical inclusion problem in classical plate theory was first attempted by Qin et al. (1991). Nevertheless, the solution provided by Qin et al. is derived as a degenerated case of the 3D linear elasticity solution, and it appears that there may be few errors in the calculation; therefore it is worthwhile revising the solution. To start with, we consider the fundamental solution of an isotropic thin plate, which is governed by the following biharmonic equation

$$D\nabla^2\nabla^2 w^G\left(\mathbf{x},\mathbf{x}'\right) = \delta(\mathbf{x}-\mathbf{x}') \tag{19}$$

where $D{:=}Eh^3/(12(1-\nu^2))$, is elastic rigidity, and ν is the Poisson ratio. Note that $\nabla^2 = \nabla^2_{x_\mu} = \nabla^2_{x'_\mu}$. The solution of Eq. (19) is well known:

$$w^G\left(\boldsymbol{x},\boldsymbol{x}'\right) = \frac{1}{8\pi D}r^2\ln r \tag{20}$$

where $r{:=}|\mathbf{x}'-\mathbf{x}| = \sqrt{(x'_1-x_1)^2+(x'_2-x_2)^2}$. Consider $r_{,\alpha}{:=}\frac{\partial r}{\partial x'_\alpha} = \cos(r,\,x'_\alpha - x_\alpha)$, $x'_\alpha - x_\alpha = rr_{,\alpha}$. It is not difficult to find that

$$k^G_{\alpha\beta} = -\frac{\partial^2 w^G}{\partial x_\alpha \partial x_\beta} = -\left\{\frac{\delta_{\alpha\beta}}{8\pi D}(2\ln r+1) + \frac{1}{4\pi D}r_{,\alpha}r_{,\beta}\right\} \tag{21}$$

$$m^G_{\alpha\beta} = -\frac{1}{4\pi}\left\{(1-\nu)r_{,\alpha}r_{,\beta} + \delta_{\alpha\beta}\left[1+\nu(\ln r+1) - \frac{(1-\nu)}{2}\right]\right\} \tag{22}$$

Subsequently,

$$m^G_{\alpha\beta,\,\gamma} = -\frac{1}{4\pi r}\left\{(1+\nu)\delta_{\alpha\beta}r_{,\gamma} + (1-\nu)\left(\delta_{\alpha\gamma}r_{,\beta} + \delta_{\beta\gamma}r_{,\alpha} - 2r_{,\alpha}r_{,\beta}r_{,\gamma}\right)\right\} \tag{23}$$

Obviously, $m^G_{\alpha\beta,\,\gamma}\to 0$, as $r\to\infty$.We now consider the thin plate's version of "*Eshelby problem*": an infinite plate with no external load, in which there is elliptical inclusion, Ω_e, embedded at the center (see Fig. 3) and there is an eigen-curvature field prescribed inside the elliptical inclusion, i.e.

$$k^*_{\alpha\beta}(\mathbf{x}) = \begin{cases} k^*_{\alpha\beta} & \forall\mathbf{x}\in\Omega_e \\ 0 & \forall\mathbf{x}\in\Omega\backslash\Omega_e \end{cases} \tag{24}$$

The equilibrium equation then takes the form

$$m_{\alpha\beta,\,\alpha\beta} + m^*_{\alpha\beta,\,\alpha\beta} = 0 \tag{25}$$

where $m^*_{\alpha\beta}$ is the induced eigen-moment, $m^*_{\alpha\beta} = -L_{\alpha\beta\zeta\eta}k^*_{\zeta\eta}$. The prescribed eigen-curvature field may suffer discontinuity across the boundary of the inclusion, and the derivatives of $k^*_{\alpha\beta}$ may be understood in a distribution sense. The integral equation for the thin plate becomes

$$\begin{aligned} w(\mathbf{x}) &- \oint_{\Gamma_\infty} m_{\alpha\beta,\,\alpha}(\mathbf{x}')n_\beta(\mathbf{x}')w^G(\mathbf{x},\mathbf{x}')\,\mathrm{d}S' + \oint_{\Gamma_\infty} m_{\alpha\beta}(\mathbf{x}')n_\beta(\mathbf{x}')w^G_{,\alpha}(\mathbf{x},\mathbf{x}')\,\mathrm{d}S' \\ &- \oint_{\Gamma_\infty} m^G_{\alpha\beta}(\mathbf{x},\mathbf{x}')w_{,\alpha}(\mathbf{x}')n_\beta(\mathbf{x}')\,\mathrm{d}S' + \oint_{\Gamma_\infty} m^G_{\alpha\beta,\,\beta}(\mathbf{x},\mathbf{x}')w(\mathbf{x}')n_\alpha(\mathbf{x}')\,\mathrm{d}S' \\ &= \int\!\!\int_{\Omega_e} k^*_{\alpha\beta}(\mathbf{x}')m^G_{\alpha\beta}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\Omega' \end{aligned} \tag{26}$$

Note that all the derivatives in Eq. (26) are taken with respect to $\mathbf{x}'$. Since $w^*_{,\alpha}$, $m^*_{\alpha\beta}$, $m^*_{\alpha\beta,\,\gamma} \to 0$, as $|\mathbf{x}| \to \infty$, it is reasonable to assume that $w_{,\alpha}$, $m_{\alpha\beta}$, $m_{\alpha\beta,\,\gamma} \to 0$ as $|\mathbf{x}| \to \infty$ By considering the behaviours of the Green's function, perturbation field, $w(\mathbf{x})$, and the eigen-deformation field at the infinity, the deflection field induced by eigen-curvature can then be described as

$$w(\mathbf{x}) = \int\!\!\int_{\Omega_e} k^*_{\alpha\beta}(\mathbf{x}')m^G_{\alpha\beta}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\Omega' = \int\!\!\int_{\Omega} k^*_{\alpha\beta}(\mathbf{x}')m^G_{\alpha\beta}(\mathbf{x}'-\mathbf{x})\,\mathrm{d}\Omega' \tag{27}$$

If the eigen-curvature is uniform inside the inclusion, we have

$$w(\mathbf{x}) = -\frac{k^*_{\alpha\beta}}{4\pi}\int\!\!\int_{\Omega_e}\left\{(1-\nu)r_{,\alpha}r_{,\beta} + \delta_{\alpha\beta}\left[(1+\nu)(\ln r + 1) - \frac{(1-\nu)}{2}\right]\right\}\mathrm{d}\Omega' \tag{28}$$

Differentation (28) and denote $r_{,\zeta} = (x'_\zeta - x_\zeta)/r = \ell_\zeta$. By using the integration scheme illustrated in Fig. 3

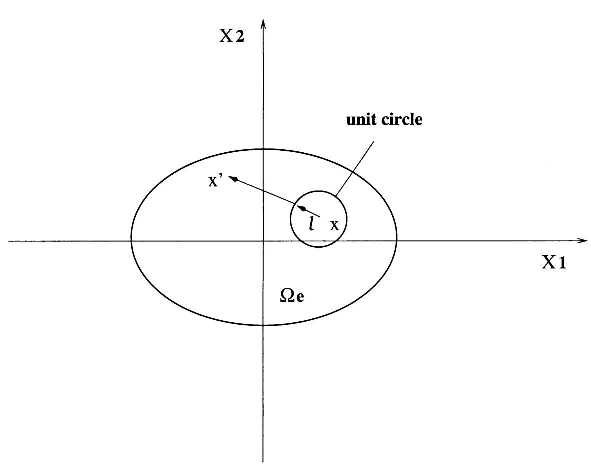


Fig. 3. The integration scheme inside an elliptical inclusion.

and choosing the polar coordinate: $d\Omega' = r\,dr\,d\theta$, Eq. (28) leads to

$$w_{,\zeta}(\mathbf{x}) = \frac{k^*_{\alpha\beta}}{4\pi}\int_0^{2\pi}\int_0^{\rho}\left\{(1-\nu)\left[\delta_{\alpha\zeta}\ell_\beta + \delta_{\beta\zeta}\ell_\alpha - 2\ell_\alpha\ell_\beta\ell_\zeta\right] + (1+\nu)\ell_\zeta\delta_{\alpha\beta}\right\} dr\,d\theta$$

where $\rho = \rho(\ell_\mu, x_\mu)$ is the root of the quadratic equation:

$$\frac{(x_1+\rho\ell_1)^2}{a_1^2} = \frac{(x_2+\rho\ell_2)^2}{a_2^2} = 1 \tag{29}$$

Solving Eq. (29), we obtain

$$\rho(\ell_\mu, x_\mu) = -\frac{f}{g} \pm \sqrt{\frac{f^{\,2}}{g^2} + \frac{e}{g}} \tag{30}$$

where $f = \lambda_\mu x_\mu$, $\lambda_\mu = \frac{\ell_\mu}{a_\mu^2}$, $g = \frac{\ell_1^2}{a_1^2} + \frac{\ell_2^2}{a_2^2}$, $e = 1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}$.Because $\pm\sqrt{f^{\,2}/g^2 + e/g}$ is an even function of $\ell(\ell_1, \ell_2)$, subsequently,

$$w_{,\zeta}(\mathbf{x}) = -\frac{k^*_{\alpha\beta}}{4\pi}\int_0^{2\pi}\left(\frac{\lambda_\mu x_\mu}{g}\right)\left\{(1-\nu)\left[\delta_{\alpha\zeta}\ell_\beta + \delta_{\beta\zeta}\ell_\alpha - 2\ell_\alpha\ell_\beta\ell_\zeta\right] + (1+\nu)\ell_\zeta\delta_{\alpha\beta}\right\} d\theta \tag{31}$$

and furthermore

$$w_{,\zeta\eta}(\mathbf{x}) = -\frac{1}{4\pi}\int_0^{2\pi}\left(\frac{\lambda_\mu\delta_{\mu\eta}}{g}\right)g_{\zeta\alpha\beta}k^*_{\alpha\beta}\,d\theta = -\frac{k^*_{\alpha b}}{4\pi}\int_0^{2\pi}\frac{\lambda_\eta g_{\zeta\alpha\beta}}{g}\,d\theta \tag{32}$$

where

$$g_{\zeta\alpha\beta} := (1-\nu)\left[\delta_{\alpha\zeta}\ell_\beta + \delta_{\beta\zeta}\ell_\alpha - 2\ell_\alpha\ell_\beta\ell_\zeta\right] + (1+\nu)\ell_\zeta\delta_{\alpha\beta} \tag{33}$$

Since $w_{,\zeta\eta} = w_{,\eta\zeta}$, we symmetrize the integral representation (32),

$$k_{\zeta\eta} = \frac{1}{2}(w_{,\zeta\eta} + w_{,\eta\zeta}) = \frac{k^*_{\alpha\beta}}{8\pi}\int_0^{2\pi}\frac{\left(\lambda_\eta g_{\zeta\alpha\beta} + \lambda_\zeta g_{\eta\alpha\beta}\right)}{g}\,d\theta \tag{34}$$

Define tensor $S_{\zeta\eta\alpha\beta}$ as

$$S_{\zeta\eta\alpha\beta} = \frac{1}{8\pi}\int_0^{2\pi}\frac{\left(\lambda_\eta g_{\alpha\beta\zeta} + \lambda_\zeta g_{\alpha\beta\eta}\right)}{g}\,d\theta \tag{35}$$

We then obtain the desired result,

$$k_{\zeta\eta} = S_{\zeta\eta\alpha\beta}k^*_{\alpha\beta} \tag{36}$$

where tensor $S_{\zeta\eta\alpha\beta}$ is the Eshelby tensor of thin plates.

Remark 2.1. *Obviously, the induced curvature inside the elliptical inclusion is constant, which is the reminiscence of Eshelby's classical result in linear elasticity, in which the induced strain inside the ellipsoidal inclusion is constant* (Eshelby, 1957). Qin et al. (1991) *had reached the same conclusion. Nevertheless, there are some differences in the expression of tensor $S_{\zeta\eta\alpha\beta}$ between present result and the* *result given by Qin et al.. In* Appendix A, *explicit expressions for every component of $S_{\zeta\eta\alpha\beta}$ are given, and they are also compared with the results in* Qin et al. (1991)

Remark 2.2. *The existence of integrals* (28), Eq. (31), *and* Eq. (32) *can be shown in a similar fashion as done by* Kellog (1953). *A discussion of general symmetrized gradient may be found in* Torquato (1997). *One of the mistakes in* Qin et al. (1991) *is that the gradient field is never symmetrized.*

3. Variational inequalities for thin plates

Over the years, the abstract structure of comparison variational principle has been developed. It departed from the original Hashin–Shtrikman formulation (Hashin and Shtrikman, 1962a, 1962b), adding the contribution from Hill (1963b), then evolving to today's Talbot and Willis formulations (Talbot and Willis, 1985; Willis, 1989), which possess both elegance in theory and generality in application. For a contemporary treatment, Talbot and Willis' formulation is adopted here to derive the variational inequalities for thin plates. As a systematic study, the elementary bounds (Voigt, 1889; Reuss, 1929) are also studied in the framework of variational inequalities. At the first sight, such extension to thin plate theory may seem to be trivial. However, we are now dealing with the variational principles of a fourth order partial differential equations, which is different from the governing equations of linear elasticity, such as how to choose prescribed boundary conditions for comparison plates. To facilitate the presentation, a few definitions are in order. Define following function spaces

$$\mathscr{W}_1(\Omega) := \left\{ w | w \in C^2(\Omega);\ w = \hat{w} \text{ and } w_n = \frac{\partial w}{\partial n} = \hat{\psi}_n,\ \forall \mathbf{x} \in \partial\Omega \right\}$$

$$\mathscr{W}_2(\Omega) := \left\{ w | w \in C^4(\Omega);\ \nabla^2\nabla^2 w = 0,\ \forall \mathbf{x} \in \partial\Omega \right\}$$

$$\mathscr{W}_3(\Omega) := \left\{ w | w \in C^2(\Omega);\ w = -\frac{1}{2}\langle k_{\alpha\beta}\rangle x_\alpha x_\beta,\ \forall \mathbf{x} \in \partial\Omega \right\}$$

$$\mathscr{M}_0\Omega := \left\{ m_{\alpha\beta} | m_{\alpha\beta} \in C^2(\Omega);\ m_{\alpha\beta,\ \alpha\beta} = 0, \right\}$$

$$\mathscr{M}_1(\Omega) := \left\{ m_{\alpha\beta} | m_{\alpha\beta} \in M_0, \text{ and } \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right) = \hat{V}_n \text{ and } M_n = \hat{M}_n,\ \forall \mathbf{x} \in \partial\Omega \right\}$$

$$\mathscr{M}_2(\Omega) := \left\{ m_{\alpha\beta} | m_{\alpha\beta} \in C^2(\Omega);\ m_{\alpha\beta} = \langle m_{\alpha\beta}\rangle \text{ and } m_{\alpha\beta,\ \beta} = 0,\ \forall \mathbf{x} \in \partial\Omega \right\}$$

$$\mathscr{K}_0(\Omega) := \left\{ k_{\alpha\beta} | k_{\alpha\beta} = -w_{,\alpha\beta}; \right\}$$

$$\mathscr{K}_1(\Omega) := \left\{ k_{\alpha\beta} | k_{\alpha\beta} = -w_{,\alpha\beta}; w \in \mathscr{W}_1(\Omega); \right\}$$

$$\mathscr{K}_2(\Omega) := \left\{ k_{\alpha\beta} | k_{\alpha\beta} = -w_{,\alpha\beta}; w \in \mathscr{W}_1(\Omega) \cap \mathscr{W}_2(\Omega); \right\}$$

$$\mathscr{K}_3(\Omega) := \{k_{\alpha\beta} | k_{\alpha\beta} = -w_{,\alpha\beta}; w \in \mathscr{W}_1(\Omega) \cap \mathscr{W}_3(\Omega);\}$$

The density of the linear elastic energy of a thin plate, $U(\mathbf{k})$ is a quadratic form:

$$U(\mathbf{k}) = \frac{1}{2} L_{\alpha\beta\zeta\eta} k_{\alpha\beta} k_{\zeta\eta} \tag{37}$$

It is evident that the potential energy density function is convex, lower semicontinuous, and proper. The moment tensor, $\mathbf{m} = m_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ can then be obtained from the constitutive relation,

$$m_{\alpha\beta} = \frac{\partial U(\mathbf{k})}{\partial k_{\alpha\beta}}, \text{ or } \mathbf{m} \in \partial_{\boldsymbol{k}} U(\mathbf{k}) \tag{38}$$

where $\partial_{\boldsymbol{k}}$ is the notation for sub-differential (e.g. Ekeland and Temam, 1976). By Fenchel–Legendre transformation, the density of complementary energy is

$$U^*(\mathbf{m}) = \sup_{\mathbf{k}} \{\mathbf{m} : \mathbf{k} - U(\mathbf{k})\} \tag{39}$$

The constitutive relations inverse to Eq. (38) is

$$k_{\alpha\beta} = \frac{\partial U^*(\mathbf{m})}{\partial m_{\alpha\beta}}, \text{ or } \mathbf{k} \in \partial_{\mathbf{m}} U^*(\mathbf{m}) \tag{40}$$

3.1. Elementary variational inequalities and elementary bounds

We begin with listing some elementary variational inequalities that are based on the principle of minimum potential energy and the principle of minimum complementary energy of thin plates. For the first type boundary condition, $\forall k_{\alpha\beta} \in \mathscr{K}_1(\Omega)$, the plate's potential energy is defined as

$$\Pi(\mathbf{k}) := \frac{1}{2} \int\int_\Omega L_{\alpha\beta\zeta\eta} k_{\alpha\beta} k_{\zeta\eta} \, \mathrm{d}\Omega \tag{41}$$

We denote $E(\mathbf{k}) = \Pi(\mathbf{k})$, if $k_{\alpha\beta} \in \mathscr{K}_2(\Omega) \subset \mathscr{K}_1(\Omega)$. The principle of minimum potential energy states that

$$E(\mathbf{k}) = \inf_{k \in \mathscr{K}_1(\Omega)} \Pi(\mathbf{k}) \tag{42}$$

Under the same type of boundary conditions, the complementary energy of the plate is

$$\begin{aligned} \Gamma(\mathbf{m}) &= \frac{1}{2} \int\int_\Omega N_{\alpha\beta\zeta\eta} m_{\alpha\beta} m_{\zeta\eta} \, \mathrm{d}\Omega - \oint_{\partial\Omega} \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right) w^0 \, \mathrm{d}S + \oint_{\partial\Omega} M_n \frac{\partial w^0}{\partial n} \, \mathrm{d}S \\ &= \frac{1}{2} \int\int_\Omega N_{\alpha\beta\zeta\eta} m_{\alpha\beta} m_{\zeta\eta} \, \mathrm{d}\Omega - \int\int_\Omega m_{\alpha\beta} k^0_{\alpha\beta} \, \mathrm{d}\Omega \end{aligned} \tag{43}$$

where $m_{\alpha\beta} \in \mathscr{M}_0(\Omega)$, $w^0 \in \mathscr{W}_1(\Omega)$.

The principle of minimum complementary energy states that

$$-E(\mathbf{k}) = \sup_{m \in \mathscr{M}_0(\Omega)} \Gamma(\mathbf{m}) \tag{44}$$

or

$$E(\mathbf{k}) = \inf_{m\epsilon\mathscr{M}_0(\Omega)} \left\{ \int\int_\Omega m_{\alpha\beta} k^0_{\alpha\beta}\, \mathrm{d}\Omega - \frac{1}{2}\int\int_\Omega N_{\alpha\beta\zeta\eta} m_{\alpha\beta} m_{\zeta\eta}\, \mathrm{d}\Omega \right\}, \quad \forall k^0_{\alpha\beta} \in \mathscr{K}_1(\Omega) \tag{45}$$

Consider the second type boundary condition (force prescribed), the overall complementary energy is

$$\Gamma(\mathbf{m}) = \frac{1}{2}\int\int_\Omega N_{\alpha\beta\zeta\eta} m_{\alpha\beta} m_{\zeta\eta}\, \mathrm{d}\Omega, \quad \forall m_{\alpha\beta} \in \mathscr{M}_1(\Omega) \tag{46}$$

The principle of minimum complementary energy states that

$$E^*(\mathbf{m}) = \inf_{m\in\mathscr{M}_1(\Omega)} \Gamma(\mathbf{m}) \tag{47}$$

where the minimizer $\mathbf{m} \in \mathscr{M}_1(\Omega)$ ensures that $k_{\alpha\beta} = N_{\alpha\beta\zeta\eta} m_{\zeta\eta}$ and $k_{\alpha\beta} = -w_{,\alpha\beta}, w \in \mathscr{W}_2(\Omega)$. Accordingly, the potential energy in this case is

$$\begin{aligned} \Pi(\mathbf{k}) &= \frac{1}{2}\int\int_\Omega L_{\alpha\beta\zeta\eta} k_{\alpha\beta} k_{\zeta\eta}\, \mathrm{d}\Omega - \oint_{\partial\Omega}\left(Q^0_n + \frac{\partial M^0_{ns}}{\partial s}\right) w\, \mathrm{d}S + \oint_{\partial\Omega} M^0_n \frac{\partial w}{\partial n}\, \mathrm{d}S \\ &= \frac{1}{2}\int\int_\Omega L_{\alpha\beta\zeta\eta} k_{\alpha\beta} k_{\zeta\eta}\, \mathrm{d}\Omega - \int\int_\Omega m^0_{\alpha\beta} k_{\alpha\beta}\, \mathrm{d}\Omega \end{aligned} \tag{48}$$

where $k_{\alpha\beta} \in \mathscr{K}_0\Omega$ and $m^0_{\alpha\beta} \in \mathscr{M}_1(\Omega)$.

The principle of minimum potential energy states that $\Pi(\mathbf{k})$ attains its minimum, when $k_{\alpha\beta} = N_{\alpha\beta\zeta\eta} m^0_{\zeta\eta}$,

$$E^*(\mathbf{m}) = \inf_{k\in\mathscr{K}_0} \left\{ \int\int_\Omega m^0_{\alpha\beta} k_{\alpha\beta}\, \mathrm{d}\Omega - \frac{1}{2}\int\int_\Omega L_{\alpha\beta\zeta\eta} k_{\alpha\beta} k_{\zeta\eta}\, \mathrm{d}\Omega \right\} \tag{49}$$

The above elementary variational inequalities have the exactly same structures as their counterparts in linear elasticity (see Hill, 1963b). Now, we are in a position to discuss the associated elementary bounds. Consider the deflection prescribed boundary condition, and let the boundary conditions compatible with the prescribed deflection value at $\delta\Omega$, i.e

$$w(\mathbf{x}) = -\frac{1}{2}\langle k_{\alpha\beta}\rangle x_\alpha x_\beta \quad \forall \mathbf{x} \in \partial\Omega \tag{50}$$

In other words, $w \in \mathscr{W}_1(\Omega) \cap \mathscr{W}_3(\Omega) =: W_d(\Omega)$. $\forall w \in W_d(\Omega)$, we make following decomposition,

$$w(\mathbf{x}) = -\frac{1}{2}\langle k_{\alpha\beta}\rangle x_\alpha x_\beta + \rho(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega \tag{51}$$

It is obvious that $\rho(\mathbf{x}) = 0$, $\forall \mathbf{x} \in \partial\Omega$, and $\frac{1}{|\Omega|}\int\int_\Omega \rho_{,\alpha\beta}\, \mathrm{d}\Omega = 0$. One may further derive that

$$\frac{1}{|\Omega|}\int\int_\Omega \rho_{,\alpha\beta}\, \mathrm{d}\Omega = \frac{1}{|\Omega|}\oint_{\partial\Omega} \rho_{,\alpha} n_\beta\, \mathrm{d}s = 0 \Longrightarrow \rho_{,\alpha} = 0, \quad \forall \mathbf{x} \in \partial\Omega \tag{52}$$

Let $k'_{\alpha\beta} = -\rho_{,\alpha\beta}$. We may define a null space of $L^2(\Omega)$ as

$$K_n(\Omega) = \left\{ k_{\alpha\beta} | k_{\alpha\beta} = -w_{,\alpha\beta},\ w \in C^2_0(\Omega)\ \mathit{and}\ \frac{1}{|\Omega|}\int\int_\Omega k_{\alpha\beta}\, \mathrm{d}\Omega = 0 \right\} \tag{53}$$

such that $k_{\alpha\beta} = \bar{k}_{\alpha\beta} + k'_{\alpha\beta}$ and $\mathbf{k}' \in K_n(\Omega)$. Following Willis (1989), by denoting $U(\mathbf{k}) = U(\bar{\mathbf{k}}, \mathbf{k}')$, the overall potential energy can then be defined as

$$\tilde{\Pi}(\bar{\mathbf{k}}) = \inf_{k'\in K_n(\Omega)} \frac{1}{|\Omega|} \int\int_{\Omega} U\left(\bar{\mathbf{k}}, \mathbf{k}'\right) \, \mathrm{d}\Omega \tag{54}$$

For linear elastic thin plate, one can identify $\tilde{\Pi}(\bar{\mathbf{k}})$ immediately,

$$\frac{1}{|\Omega|} \int\int_{\Omega} U\left(\bar{\mathbf{k}}, \mathbf{k}'\right) \mathrm{d}\Omega = \frac{1}{2|\Omega|} \int\int_{\Omega} L_{\alpha\beta\zeta\eta}\left(\bar{k}_{\alpha\beta} + k'_{\alpha\beta}\right)\left(\bar{k}_{\zeta\eta} + k'_{\zeta\eta}\right) \mathrm{d}\Omega \geq \frac{1}{2|\Omega|} \int\int_{\Omega} L_{\alpha\beta\zeta\eta}\bar{k}_{\alpha\beta}\bar{k}_{\zeta\eta} \, \mathrm{d}\Omega$$
$$= \frac{1}{2}\langle m_{\alpha\beta}\rangle \, \langle k_{\alpha\beta}\rangle \tag{55}$$

As a matter of fact, $\forall w \in \mathscr{W}_1(\Omega) \cap \mathscr{W}_3(\Omega)$, and $\forall m_{\alpha\beta} \in \mathscr{M}_1(\Omega) \cap \mathscr{M}_2(\Omega)$, by Lemma 2.4 one may show that

$$\langle \mathbf{m}{:}\mathbf{k}\rangle = \frac{1}{|\Omega|} \int\int_{\Omega} \mathbf{m}{:}\mathbf{k} \, d\Omega = \langle \mathbf{m}\rangle{:}\langle \mathbf{k}\rangle \tag{56}$$

Then the overall potential energy under the prescribed deflection boundary condition is

$$\tilde{\Pi}(\bar{\mathbf{k}}) = \frac{1}{2}\langle \mathbf{m}\rangle{:}\langle \mathbf{k}\rangle \tag{57}$$

The inequality (42) can then be modified as

$$\tilde{\Pi}(\bar{\mathbf{k}}) = \inf_{k'\in K_n(\Omega)} \Pi\left(\bar{\mathbf{k}}, \mathbf{k}'\right) \leq \overline{\Pi(\bar{\mathbf{k}})} = \sum_{r=1}^{n+1} c_r \Pi_r(\bar{\mathbf{k}}) \tag{58}$$

where $c_r = h|\Omega_r|/h|\Omega| = |\Omega_r|/|\Omega|$, and $\Pi_r(\bar{\mathbf{k}}) := \int\int_{\Omega r} U(\bar{\mathbf{k}}) \, \mathrm{d}\Omega$. Eq. (58) is the well-known Voigt bound.

For the second type boundary problem, the prescribed force boundary conditions $(Q_n + \partial M_{ns}/\partial_s) = \hat{V}_n$ and $M_n = \hat{M}_n$ is chosen such that they are compatible with the special boundary value of moments:

$$m_{\alpha\beta} = \langle m_{\alpha\beta}\rangle, \qquad m_{\alpha\beta,\,\beta} = 0, \quad \forall \mathbf{x} \in \partial\Omega$$

For $m_{\alpha\beta} \in \mathscr{M}_1(\Omega) \cap \mathscr{M}_2(\Omega)$, we make the decomposition

$$m_{\alpha\beta} = \langle m_{\alpha\beta}\rangle + \tau_{\alpha\beta} \tag{59}$$

By virtue of Lemma 2.3, it can be readily shown that

$$\tau_{\alpha\beta} = 0, \qquad \tau_{\alpha\beta,\,\beta} = 0, \quad \forall x \in \partial\Omega \Longrightarrow \frac{1}{|\Omega|} \int\int_{\Omega} \tau_{\alpha\beta} \, \mathrm{d}\Omega = 0 \tag{60}$$

and furthermore, $\tau_{\alpha\beta} \in M_n$ where

$$M_n(\Omega); = \left\{ m_{\alpha\beta} | m_{\alpha\beta,\,\alpha\beta} = 0, \frac{1}{|\Omega|} \int\int_{\Omega} m_{\alpha\beta} \, \mathrm{d}\Omega = 0 \right\} \tag{61}$$

Accordingly, one can define the overall complementary energy potential as

$$\tilde{\Gamma}(\bar{\mathbf{m}}) := \inf_{\boldsymbol{\tau}\in M_n(\Omega)} \Gamma(\bar{\mathbf{m}}, \boldsymbol{\tau}) \tag{62}$$

Again, for linear elastic plate, we can identify $\tilde{\Gamma}(\bar{\mathbf{m}})$ as

$$\Gamma(\bar{\mathbf{m}};\boldsymbol{\tau}) = \frac{1}{2}\int\int_{\Omega}\left(\bar{m}_{\alpha\beta}+\tau_{\alpha\beta}\right)N_{a\beta\zeta\eta}\left(\bar{m}_{\zeta\eta}+\tau_{\zeta\eta}\right)\,\mathrm{d}\Omega \geq \frac{1}{2}\int\int_{\Omega}\left(N_{\alpha\beta\zeta\eta}\bar{m}_{\alpha\beta}\bar{m}_{\zeta\eta}\right)\,\mathrm{d}\Omega = \frac{1}{2}\langle m_{\alpha\beta}\rangle\,\langle k_{\alpha\beta}\rangle \tag{63}$$

The variational inequality (62) furnishes the estimate

$$\tilde{\Gamma}(\bar{\mathbf{m}}) := \inf_{\boldsymbol{\tau}\in M_n(\Omega)}\Gamma(\bar{\mathbf{m}},\boldsymbol{\tau}) \leq \overline{\Gamma(\bar{\mathbf{m}})} = \sum_{r=1}^{n+1}c_r\Gamma_r(\bar{\mathbf{m}}) \tag{64}$$

where $\Gamma_r(\bar{\mathbf{m}}) = \int\int_{\Omega r}U^*(\bar{\mathbf{m}})\,\mathrm{d}\Omega$ Eq. (64) is the well-known Reuss bound.

3.2. The Hashin-Shtrikman/Talbot–Willis type principles

Consider the first type (deflection prescribed) boundary-value problem. Let $w \in \mathscr{W}_1(\Omega)$ be a special kinematic admissible deflection field, which is the superposition:

$$w(\mathbf{x}) = w^0(\mathbf{x}) + w^1(\mathbf{x}) \tag{65}$$

such that $w^0(\mathbf{x})$ is a solution of the first type boundary-value problem in the comparison plate, which has the elastic stiffness, $L^0_{\alpha\beta\zeta\eta}$; and $w^1(\mathbf{x}) \in C^2_0(\Omega)$. Accordingly,

$$k_{\alpha\beta} = k^0_{\alpha\beta} + k^1_{\alpha\beta} \tag{66}$$

where $\{k^1_{\alpha\beta}\}:=\{-w^1_{,\alpha\beta}\} \in B$, the closed subspace of $[L^2(\Omega)]^3$. In addition that $(w^0, k^0_{\alpha\beta})$ is kinematically admissible, for the comparison plate, we have: $m^0_{\alpha\beta} = L^0_{\alpha\beta\zeta\eta}k^0_{\zeta\eta}$, $m^0_{\alpha\beta,\,\alpha\beta} = 0$. We are looking for the solution of the following optimization problem:

$$\text{(The primal problem)}\quad \mathscr{P}\colon \inf_{\mathbf{k}^1\in B}\Pi(\mathbf{k}^1) \tag{67}$$

where

$$\Pi(\mathbf{k}^1) := \int\int_{\Omega}U\left(\mathbf{k}^0+\mathbf{k}^1\right)\,\mathrm{d}\Omega = \frac{1}{2}\int\int_{\Omega}L_{\alpha\beta\zeta\eta}\left(k^0_{\alpha\beta}+k^1_{\alpha\beta}\right)\left(k^0_{\zeta\eta}+k^1_{\zeta\eta}\right)\,\mathrm{d}\Omega \tag{68}$$

Let $\mathbf{m}^* \in B^*$, the dual space of B. Define the dual potential $\Pi^*(\mathbf{m}^*)$,

$$\Pi^*(\mathbf{m}^*) = \sup_{\mathbf{k}^1\in B}\left\{\left(\mathbf{m}^*,\mathbf{k}^1\right) - \Pi\left(\mathbf{k}^1\right)\right\}. \tag{69}$$

We say, $\mathbf{m}^* \in B^0$, the set of annihilators of B, if

$$\left(\boldsymbol{m}^*,\mathbf{k}^1\right) = \int\int_{\Omega}m^*_{\alpha\beta}k^1_{\alpha\beta}\,\mathrm{d}\Omega = 0 \tag{70}$$

which posts additional constraints on $\mathbf{m}^*$ (see Lemma 2.1). Subsequently,

$$\Pi(\mathbf{k}^1) + \Pi^*(\mathbf{m}^*) \geq 0,\quad \forall\mathbf{m}^* \in B^0 \tag{71}$$

it then implies

$$-\Pi^*(\mathbf{m}^*) \le \inf_{\mathbf{k}^1 \in B} \mathscr{P} \le \Pi(\mathbf{k}^1) \tag{72}$$

This also suggests a dual problem

$$\text{(The dual problem)} \quad \mathscr{P}^*\colon \sup_{\mathbf{m}^* \in B^0} \left\{ -\Pi^*(\mathbf{m}^*) \right\}. \tag{73}$$

For practical purpose, instead of optimizing the dual problem (73) of the deflection prescribed boundary-value problem, it is often convenient to consider a somewhat equivalent problem: optimize the complementary potential energy under the force prescribed boundary-value problem. Define

$$\Gamma(\mathbf{m}^1) = \int\int_\Omega U^*(\mathbf{m}^0 + \mathbf{m}^1)\, \mathrm{d}\Omega = \frac{1}{2}\int\int_\Omega N_{\alpha\beta\zeta\eta}\left(m^0_{\alpha\beta} + m^1_{\alpha\beta}\right)\left(m^0_{\zeta\eta} + m^1_{\zeta\eta}\right) \mathrm{d}\Omega \tag{74}$$

where $\mathbf{m} = \mathbf{m}^0 + \mathbf{m}^1$ is a special statically admissible moment field; in which $\mathbf{m}^0$ is the solution of the second type boundary problem in the comparison plate, and $\mathbf{m}^1$ is the perturbation, such that $\mathbf{m}^1 \in H(\Omega) \subset [L^2_0(\Omega)]^3$.

Consider the optimization problem

$$\mathscr{P}_d\colon \inf_{\mathbf{m}^1 \in H(\Omega)} \Gamma(\mathbf{m}^1) \tag{75}$$

It has a duality approach too, i.e. $\forall \mathbf{k}^* \in H^0$, the annihilator set of H, i.e.

$$H^0 := \left\{ \mathbf{k}^* \int\int_\Omega m^1_{\alpha\beta} k^*_{\alpha\beta}\, \mathrm{d}\Omega = 0, \quad \forall \mathbf{m}^1 \in H \right\} \tag{76}$$

The restriction on $k^*_{\alpha\beta}$ has been stated in Lemma 2.2. Consequently, we have

$$-\Gamma^*(\mathbf{k}^*) \le \inf_{\mathbf{m}^1 \in H(\Omega)} \mathscr{P}_d \le \Gamma(\mathbf{m}^1) \tag{77}$$

where

$$\Gamma^*(\mathbf{k}^*) := \sup_{\mathbf{m}^1 \in H} \left\{ \left(\mathbf{m}^1, \mathbf{k}^*\right) - \Gamma(\mathbf{m}^1) \right\} \tag{78}$$

Remark 3.1. *The conditions that define the annihilator sets of B and H*, (70) *and* (76) *have important physical interpretations*. Eq. (70) *is related to the virtual work identity*

$$\int\int_\Omega m^*_{\alpha\beta} k^1_{\alpha\beta}\, \mathrm{d}\Omega = \int\int_\Omega m^*_{\alpha\beta}\left(k_{\alpha\beta} - k^0_{\alpha\beta}\right) \mathrm{d}\Omega = 0 \tag{79}$$

and (76) *is related to the virtual complementary work*

$$\int\int_\Omega k^*_{\alpha\beta} m^1_{\alpha\beta}\, \mathrm{d}\Omega = \int\int_\Omega k^*_{\alpha\beta}\left(m_{\alpha\beta} - m^0_{\alpha\beta}\right) \mathrm{d}\Omega = 0 \tag{80}$$

Now, we are in the position to derive the Hashin–Shtriman/Talbot–Willis type principle. Introduce two comparison of functionals

$$\Pi_0(\mathbf{k}^1) := \frac{1}{2}\int\int_{\Omega} U_0(\mathbf{k}^0 + \mathbf{k}^1)\,\mathrm{d}\Omega = \frac{1}{2}\int\int_{\Omega} L^0_{\alpha\beta\zeta\eta}\left(\mathbf{k}^0_{\alpha\beta} + \mathbf{k}^1_{\alpha\beta}\right)\left(\mathbf{k}^0_{\zeta\eta} + \mathbf{k}^1_{\zeta\eta}\right)\mathrm{d}\Omega,\ \text{if } L_{\alpha\beta\zeta\eta} - L^0_{\alpha\beta\zeta\eta} > 0$$

$$\Pi_0(\mathbf{k}^1) := \frac{1}{2}\int\int_{\Omega} U_0(\mathbf{k}^0 + \mathbf{k}^1)\,\mathrm{d}\Omega = \frac{1}{2}\int\int_{\Omega} L^0_{\alpha\beta\zeta\eta}\left(\mathbf{k}^0_{\alpha\beta} + \mathbf{k}^1_{\alpha\beta}\right)\left(\mathbf{k}^0_{\zeta\eta} + \mathbf{k}^1_{\zeta\eta}\right)\mathrm{d}\Omega,\ \text{if } L_{\alpha\beta\zeta\eta} - L^0_{\alpha\beta\zeta\eta} < 0$$

and the associated potential differences

$$\underline{F}(\mathbf{k}^1) := \Pi(\mathbf{k}^1) - \Pi_0(\mathbf{k}^1), \qquad \Delta L_{\alpha\beta\zeta\eta} := L_{\alpha\beta\zeta\eta} - L^0_{\alpha\beta\zeta\eta} > 0 \tag{81}$$

$$\bar{F}(\mathbf{k}^1) := \Pi(\mathbf{k}^1) - \Pi^0(\mathbf{k}^1), \qquad \Delta L_{\alpha\beta\zeta\eta} := L_{\alpha\beta\zeta\eta} - L^0_{\alpha\beta\zeta\eta} < 0 \tag{82}$$

Then define the moment polarization tensor $\tau_{\alpha\beta}$, such that

$$m_{\alpha\beta} = L^0_{\alpha\beta\zeta\eta}k_{\zeta\eta} + \tau_{\alpha\beta},\ \text{or}\ \tau_{\alpha\beta} = \left(L_{\alpha\beta\zeta\eta} - L^0_{\alpha\beta\zeta\eta}\right)k_{\zeta\eta} \tag{83}$$

By the transformations,

$$\underline{F}^*(\boldsymbol{\tau}) = \sup_{\mathbf{k}^1\in B}\left\{\left(\boldsymbol{\tau}, \mathbf{k}^1\right) - \underline{F}(\mathbf{k}^1)\right\} \tag{84}$$

$$\bar{F}^*(\boldsymbol{\tau}) = \inf_{\mathbf{k}^1\in B}\left\{\left(\boldsymbol{\tau}, \mathbf{k}^1\right) - \bar{F}(\mathbf{k}^1)\right\} \tag{85}$$

Note that the supremum in Eq. (84) and infimum in Eq. (85) are attained when Eq. (83) holds. It can be readily shown that the primal problem can be realized by the following comparison strategy,

$$\inf_{\mathbf{k}^1\in B}\left\{\left(\boldsymbol{\tau}, \mathbf{k}^1\right) + \Pi_0(\mathbf{k}^1)\right\} - \underline{F}^*(\boldsymbol{\tau}) \le \inf\mathscr{P} \le \inf_{\mathbf{k}^1\in B}\left\{\left(\boldsymbol{\tau}, \mathbf{k}^1\right) + \Pi^0(\mathbf{k}^1)\right\} - \bar{F}^*(\boldsymbol{\tau}) \tag{86}$$

Substituting $k^1_{\alpha\beta} = \Delta L^{-1}_{\alpha\beta\zeta\eta}\tau_{\zeta\eta} - k^0_{\alpha\beta}$ into Eqs. (84) and (85), we have

$$\begin{aligned}
&\underline{F}^*(\boldsymbol{\tau})\left(\text{or } \bar{F}^*(\boldsymbol{\tau})\right)\\
&= \int\int_{\Omega}\left\{\tau_{\alpha\beta}k_{\alpha\beta} - \frac{1}{2}\Delta L_{\alpha\beta\zeta\eta}k^1_{\alpha\beta}k^1_{\alpha\beta} - \frac{1}{2}\Delta L_{\alpha\beta\zeta\eta}k^0_{\alpha\beta}k^0_{\zeta\eta} - L_{\alpha\beta\zeta\eta}k^0_{\alpha\beta}k^1_{\zeta\eta} + L^0_{\alpha\beta\zeta\eta}k_{\alpha\beta}k^1_{\zeta\eta}\right\}\mathrm{d}\Omega\\
&= \frac{1}{2}\int\int_{\Omega}\left(\Delta L^{-1}_{\alpha\beta\zeta\eta}\tau_{\alpha\beta}\tau_{\zeta\eta} - \tau_{\alpha\beta}k^0_{\alpha\beta}\right)\mathrm{d}\Omega
\end{aligned} \tag{87}$$

Compute

$$\underline{I} := \inf_{\mathbf{k}^1\in B}\left\{\left(\boldsymbol{\tau}, \mathbf{k}^1\right) + \Pi_0(\mathbf{k}^1)\right\} - \underline{F}^*(\boldsymbol{\tau}) \tag{88}$$

$$\bar{I} := \inf_{\mathbf{k}^1\in B}\left\{\left(\boldsymbol{\tau}, \mathbf{k}^1\right) + \Pi_0(\mathbf{k}^1)\right\} - \underline{F}^*(\boldsymbol{\tau}) \tag{89}$$

The infima are attained when (recall and compare with the Remark 3.1)

$$\int\int_{\Omega}\Big(L^0_{\alpha\beta\zeta\eta}k_{\zeta\eta}+\tau_{\alpha\beta}\Big)k^1_{\alpha\beta}\,\mathrm{d}\Omega=0 \tag{90}$$

or the subsidiary condition if the functions involved are sufficiently smooth

$$\Big(L^0_{\alpha\beta\zeta\eta}k^1_{\zeta\eta}+\tau_{\alpha\beta}\Big)_{,\alpha\beta}=0 \tag{91}$$

One will find that

$$\underline{I}(\text{or }\bar{I})=\int\int_{\Omega}\left\{U_0(\mathbf{k}^0)-\frac{1}{2}\Big(\Delta L^{-1}_{\alpha\beta\zeta\eta}\tau_{\alpha\beta}\tau_{\zeta\eta}-\tau_{\alpha\beta}k^1_{\alpha\beta}-2\tau^0_{\alpha\beta}k^0_{\alpha\beta}\Big)\right\}\mathrm{d}\Omega \tag{92}$$

We just showed that Eq. (86) is exactly the same as the Hashin–StriLman–Hill type inequality:

$$\int\int_{\Omega}\big(U(\mathbf{k})-U_0(\mathbf{k}^0)\big)\,\mathrm{d}\Omega-\int\int_{\Omega}\left\{\frac{1}{2}\Big(\Delta L^{-1}_{\alpha\beta\zeta\eta}\tau_{\alpha\beta}\tau_{\zeta\eta}-\tau_{\alpha\beta}k^1_{\alpha\beta}-2\tau_{\alpha\beta}k^0_{\alpha\beta}\Big)\right\}\mathrm{d}\Omega \tag{93}$$

The direction of the inequality depends on the positive definiteness or negative definiteness of the tensor $\Delta L^{-1}_{\alpha\beta\zeta\eta}$. Similarly, consider the prescribed moment boundary-value problem and introduce two comparison functionals

$$\Gamma_0(\mathbf{m}^1)=\frac{1}{2}\int\int_{\Omega}N^0_{\alpha\beta\zeta\eta}\Big(m^0_{\alpha\beta}+m^1_{\alpha\beta}\Big)\Big(m^0_{\zeta\eta}+m^1_{\zeta\eta}\Big)\,\mathrm{d}\Omega,\quad \text{if } N_{\alpha\beta\zeta\eta}-N^0_{\alpha\beta\zeta\eta}>0$$

$$\Gamma_0(\mathbf{m}^1)=\frac{1}{2}\int\int_{\Omega}N^0_{\alpha\beta\zeta\eta}\Big(m^0_{\alpha\beta}+m^1_{\alpha\beta}\Big)\Big(m^0_{\zeta\eta}+m^1_{\zeta\eta}\Big)\,\mathrm{d}\Omega,\quad \text{if } N_{\alpha\beta\zeta\eta}-N^0_{\alpha\beta\zeta\eta}<0$$

and form the potential differences

$$\underline{G}(\mathbf{m}^1):=\Gamma(\mathbf{m}^1)-\Gamma_0(\mathbf{m}^1) \tag{94}$$

$$\bar{G}(\mathbf{m}^1):=\Gamma(\mathbf{m}^1)-\Gamma_0(\mathbf{m}^1) \tag{95}$$

Recall that $\mathbf{m}^0$ is the solution of the prescribed moment boundary-value problem and $\mathbf{m}^1$ belongs to a closed subspace of $[L^2(\Omega)]^3$. By defining "the polarization curvature" as

$$\eta_{\alpha\beta}:=\Big(N_{\alpha\beta\zeta\eta}-N^0_{\alpha\beta\zeta\eta}\Big)m_{\zeta\eta},\quad \text{or}\quad k_{\alpha\beta}=N^0_{\alpha\beta\zeta\eta}m_{\zeta\eta}+\eta_{\alpha\beta} \tag{96}$$

the dual potential differences can be realized as

$$\underline{G}^*(\boldsymbol{\eta}):=\sup_{\mathbf{m}^1\in H}\left\{\big(\boldsymbol{\eta},\mathbf{m}^1\big)-\underline{G}(\mathbf{m}^1)\right\} \tag{97}$$

$$\bar{G}^*(\boldsymbol{\eta}):=\inf_{\mathbf{m}^1\in H}\left\{\big(\boldsymbol{\eta},\mathbf{m}^1\big)-\bar{G}(\mathbf{m}^1)\right\} \tag{98}$$

which ensures the following estimate of optimization problem (75)

$$\inf_{\mathbf{m}^1\in H}\left\{\big(\boldsymbol{\eta},\mathbf{m}^1\big)+\Gamma_0(\mathbf{m}^1)\right\}-\underline{G}^*(\boldsymbol{\eta})\leq \inf P_d\leq \inf_{\mathbf{m}^1\in H}\left\{\big(\boldsymbol{\eta},\mathbf{m}^1\big)+\Gamma^0(\mathbf{m}^1)\right\}-\bar{G}^*(\mathbf{m}^1) \tag{99}$$

Substituting $m^1_{\alpha\beta} = \Delta N^{-1}_{\alpha\beta\zeta\eta}\eta_{\zeta\eta} - m^0_{\alpha\beta}$ into Eqs. (97) and (98), the supremum and infimum will be attained, and it yields

$$\underline{G}^*(\boldsymbol{\eta})\Big(\text{or } \bar{G}^*(\boldsymbol{\eta})\Big) \int\int_\Omega \left\{ \frac{1}{2}\Delta N^{-1}_{\alpha\beta\zeta\eta}\eta_{\alpha\beta}\eta_{\zeta\eta} - \eta_{\alpha\beta}m^0_{\alpha\beta} \right\} \mathrm{d}\Omega \tag{100}$$

Compute

$$\underline{J} := \inf_{\mathbf{m}^1 \in H} \left\{ \left(\boldsymbol{\eta}, \mathbf{m}^1\right) + \Gamma_0(\mathbf{m}^1) \right\} - \bar{G}^*(\boldsymbol{\eta}) \tag{101}$$

$$\bar{J}: \inf_{\mathbf{m}^1 \in H} \left\{ \left(\boldsymbol{\eta}, \mathbf{m}^1\right) + \Gamma^0(\mathbf{m}^1) \right\} - \bar{G}^*(\boldsymbol{\eta}) \tag{102}$$

Again, recall Remark 3.1 the infima can be reached if the principle of virtual complementary work is applied

$$\int\int_\Omega \left(N_{\alpha\beta\zeta\eta}m_{\zeta\eta} + \eta_{\alpha\beta}\right)m^1_{\alpha\beta}\,\mathrm{d}\Omega = 0 \tag{103}$$

which is equivalent to the following subsidiary conditions if the functions involved are sufficiently smooth,

$$m^1_{\alpha\beta,\,\alpha\beta} = 0, \qquad \in_{\alpha\gamma} k^c_{\alpha\beta,\,\beta\gamma} = 0 \tag{104}$$

where $k^c_{\alpha\beta} = N^0_{\alpha\beta\zeta\eta}m^1_{\zeta\eta} + \eta_{\alpha\beta}$.It is not difficult to find that

$$\underline{J}(\text{or } \bar{J}) = \int\int_\Omega \left\{ U^*(\mathbf{m}^0) - \frac{1}{2}\left(\Delta N^{-1}_{\alpha\beta\zeta\eta}\eta_{\alpha\beta}\eta_{\zeta\eta} - \eta_{\alpha\beta}m^1_{\alpha\beta} - 2\eta_{\alpha\beta}m^0_{\alpha\beta}\right) \right\} \mathrm{d}\Omega \tag{105}$$

The variational estimates (99) take the form,

$$\int\int_\Omega \left(U^*(\mathbf{m}) - U^*_0(\mathbf{m})\right)^0 \mathrm{d}\Omega - \frac{1}{2}\int\int_\Omega \left\{ \Delta N^{-1}_{\alpha\beta\zeta\eta}\eta_{\alpha\beta}\eta_{\zeta\eta} - \eta_{\alpha\beta}m^1_{\alpha\beta} - 2\eta_{\alpha\beta}m^0_{\alpha\beta} \right\} \mathrm{d}\Omega \tag{106}$$

4. Overall elastic stiffness estimate

In this section, the elastic stiffness/compliance of suitable composite plates are estimated. Consider isotropic case:

$$L_{\alpha\beta\zeta\eta} = \frac{D(1-\nu)}{2}\left(\delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\zeta}\right) + D\nu\delta_{\alpha\beta}\delta_{\zeta\eta} = D(1-\nu)I_{\alpha\beta\zeta\eta} + 2D\nu J_{\alpha\beta\zeta\eta} \tag{107}$$

$$N_{\alpha\beta\zeta\eta} = \frac{(1-\nu)}{2D(1-\nu^2)}\left(\delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\zeta}\right) - \frac{\nu}{D(1-\nu^2)}\delta_{\alpha\beta}\delta_{\zeta\eta} = \frac{1}{D(1-\nu)}I_{\alpha\beta\zeta\eta} - \frac{1}{D(1-\nu^2)}J_{\alpha\beta\zeta\eta} \tag{108}$$

where $I_{\alpha\beta\zeta\eta} := \frac{1}{2}(\delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\zeta})$, $J_{\alpha\beta\zeta\eta} := \frac{1}{2}\delta_{\alpha\beta}\delta_{\zeta\eta}$. Define $K_{\alpha\beta\zeta\eta} := \frac{1}{2}(\delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\zeta} - \delta_{\alpha\beta}\delta_{\zeta\eta})$. Thus, $\mathbf{I} = \mathbf{J} + \mathbf{K}$ and $\mathbf{J}\cdot\mathbf{J} = \mathbf{J}$, $\mathbf{J}\cdot\mathbf{K} = \mathbf{K}\cdot\mathbf{J} = 0$, $\mathbf{K}\cdot\mathbf{K} = \mathbf{K}$. The elastic stiffness and compliance tensors can then be put into canonical forms

$$L_{\alpha\beta\zeta\eta} = 2\kappa_\rho J_{\alpha\beta\zeta\eta} + 2\mu_p K_{\alpha\beta\zeta\eta} \tag{109}$$

$$N_{\alpha\beta\zeta\eta} = \frac{1}{2\kappa_p} J_{\alpha\beta\zeta\eta} + \frac{1}{2\mu_\rho} K_{\alpha\beta\zeta\eta} \tag{110}$$

where

$$\kappa_p := \frac{D_p(1+\nu)}{2} = \frac{Eh^3}{24(1-\nu)} \tag{111}$$

$$\mu_p := \frac{D_p(1-\nu)}{2} = \frac{Eh^3}{24(1+\nu)} \tag{112}$$

$$D_p = \kappa_p + \mu_p, \tag{113}$$

$$\nu = \frac{\kappa_p - \mu_p}{\kappa_p + \mu_p} \tag{114}$$

4.1. Hashin–Shiribman type upper/lower bounds

It can be shown that (see Appendix B)

$$\mathbf{k}^1 = -\Sigma\boldsymbol{\tau} = -\int\int_\Omega \Sigma^\infty(\mathbf{x}' - \mathbf{x})[\boldsymbol{\tau} - \langle\boldsymbol{\tau}\rangle](\mathbf{x}')\,\mathrm{d}\Omega' \tag{115}$$

Hence Eq. (93) takes the form

$$2(\Pi_0 - \Pi)\big(\boldsymbol{\tau}, \Delta L^{-1}\boldsymbol{\tau}\big) + (\boldsymbol{\tau}, \Sigma\boldsymbol{\tau}) - 2\big(\boldsymbol{\tau}, \mathbf{k}^0\big) \tag{116}$$

which has the same form as the linear elasticity solutions, except that stress polarization becomes moment polarization and strain perturbation becomes curvature perturbation. Following standard procedure (again readers are referred to Willis, 1977, 1981 or Walpole, 1966a, 1981), one should have no difficulty to derive that

$$\bar{\mathbf{k}}(\mathbf{L} - \bar{\mathbf{L}})\bar{\mathbf{k}} \leqslant 0 \tag{117}$$

where $\bar{\mathbf{L}} = \sum_{r=1}^{n} c_r \mathbf{L}_r \mathbf{A}_r \left(\sum_{r=1}^{n} c_r \mathbf{A}_r\right)^{-1}$, $\mathbf{A}_r = [\mathbf{I} + \mathbf{P}_0(\mathbf{L}_r - \mathbf{L}_0)]^{-1}$ For macroscopically isotropic plate (see Appendix B)

$$\mathbf{P}_0^{-1} = \mathbf{L}_0 + \mathbf{L}_0^* \tag{118}$$

and $\mathbf{L}_0^* = 2\kappa_p^*\mathbf{J} + 2\mu_p^*\mathbf{K}$ with $\kappa_p^* = \mu_p^0; \mu_p^* = 2\kappa_p^0 + \mu_p^0$.

Remark 4.1. *By using the variational inequalities* (106), *one can also derive that*

$$\bar{\mathbf{m}}(\mathbf{N} - \bar{\mathbf{N}})\bar{\mathbf{m}} \geqslant 0 \tag{119}$$

where $\bar{\mathbf{N}} = \sum_{r=1}^{n} c_r \mathbf{N}_r \mathbf{B}_r \left(\sum_{r=1}^{n} c_r \mathbf{B}_r\right)^{-1}$, $\mathbf{B}_r = [\mathbf{I} + \mathbf{Q}_0(\mathbf{N}_r - \mathbf{N}_0)]^{-1}$ with $\mathbf{Q}_0 = \mathbf{N}_0 + \mathbf{N}_0^*$ and $\mathbf{N}_0^* = \frac{1}{2\kappa_p^*}\boldsymbol{J} + \frac{1}{2\mu_p^*}\mathbf{K}$.

Let

$$\kappa_g := \max_{1\le r\le n}\left\{\kappa_p^{(r)}\right\}, \qquad \kappa_g^* = \mu_g$$

$$\kappa_l := \min_{1\le r\le n}\left\{\kappa_p^{(r)}\right\}, \qquad \kappa_l^* = \mu_l$$

$$\mu_g := \max_{1\le r\le n}\left\{\mu_p^{(r)}\right\}, \qquad \mu_g^* = 2\kappa_g + \mu_g$$

$$\mu_l := \min_{1\le r\le n}\left\{\mu_p^{(r)}\right\}, \qquad \mu_l^* = 2\kappa_l + \mu_l \tag{120}$$

Since (see Walpole, 1969),

$$\bar{\mathbf{L}} = \sum_{r=1}^{n} c_r \mathbf{L}_r \mathbf{A}_r \left(\sum_{r=1}^{n} c_r \mathbf{A}_r\right)^{-1} = \left[\sum_{r=1}^{n} c_r \left[\mathbf{L}_r + \mathbf{L}_0^*\right]^{-1}\right] - \mathbf{L}_0^* \tag{121}$$

it readily yields

$$\left[\sum_{r=1}^{n} c_r\left(\kappa_l^* + \kappa_p^{(r)}\right)^{-1}\right] - \kappa_l^* \le \kappa_p \le \left[\sum_{r=1}^{n} c_r\left(\kappa_g^* + \kappa_p^{(r)}\right)^{-1}\right] - \kappa_g^* \tag{122}$$

$$\left[\sum_{r=1}^{n} c_r\left(\mu_l^* + \mu_p^{(r)}\right)^{-1}\right] - \mu_l^* \le \mu_p \le \left[\sum_{r=1}^{n} c_r\left(\mu_g^* + \mu_p^{(r)}\right)^{-1}\right] - \mu_g^* \tag{123}$$

The above results resemble the classical results in micromechanics of linear elasticity (Hashin and Shtrikman, 1962b; Walpole, 1969) in character, but having different physical contents as well as numerical quantities. By adding Eqs. (122), (123) and utilizing Eq. (113), an explicit estimate for the thin plate's rigidity is obtained

$$\left[\sum_{r=1}^{n} c_r\left(\kappa_l^* + \kappa_p^{(r)}\right)^{-1}\right] + \left[\sum_{r=1}^{n} c_r\left(\mu_l^* + \mu_p^{(r)}\right)^{-1}\right] - 2D_l \le D_p \le \left[\sum_{r=1}^{n} c_r\left(\kappa_g^* + \kappa_p^{(r)}\right)^{-1}\right] + \left[\sum_{r=1}^{n} c_r\left(\mu_g^* + \mu_p^{(r)}\right)^{-1}\right] - 2D_g \tag{124}$$

Following Hill (1965), we denote

$$\alpha_p := \frac{\kappa_p}{\kappa_p + \kappa_p^*} \tag{125}$$

$$\beta_p := \frac{\mu_p}{\mu_p + \mu_p^*} \tag{126}$$

One may verify that

$$\alpha_p = 1 - 2\beta_p; \qquad \alpha_p = \frac{1+\nu}{2}; \qquad \beta_p = \frac{1}{4}(1-\nu) \tag{127}$$

$$-1 < \nu \leq \frac{1}{2} \Longrightarrow 0 < \alpha_p \leq \frac{3}{4}; \qquad \frac{1}{8} < \beta_p \leq \frac{1}{2} \tag{128}$$

It is interesting to compare them with their counterparts in linear elasticity:

$$\alpha_{\rm e} = \frac{1+\nu}{3(1-\nu)} \tag{129}$$

$$\beta_{\rm e} = \frac{2(4-5\nu)}{15(1-\nu)} \tag{130}$$

For the thin plates that are made of two phase composite materials, assuming $\kappa_p^{(1)} - \kappa_p^{(2)} > 0$, $\mu_p^{(1)} - \mu_p^{(2)} > 0$ one will have

$$\kappa_p^{(2)} + \frac{c_1\left(\kappa_p^{(1)} - \kappa_p^{(2)}\right)}{1 + c_2\alpha_{p2}\left(\kappa_p^{(1)}/\kappa_p^{(2)} - 1\right)} \leq \kappa_p \leq \kappa_p^{(1)} + \frac{c_2\left(\kappa_p^{(2)} - \kappa_p^{(1)}\right)}{1 + c_1\alpha_{p1}\left(\kappa_p^{(2)}/\kappa_p^{(1)} - 1\right)} \tag{131}$$

$$\mu_p^{(2)} + \frac{c_1\left(\mu_p^{(1)} - \mu_p^{(2)}\right)}{1 + c_2\beta_{p2}\left(\mu_p^{(1)}/\mu_p^{(2)} - 1\right)} \leq \mu_p \leq \mu_p^{(1)} + \frac{c_2\left(\mu_p^{(2)} - \mu_p^{(1)}\right)}{1 + c_1\beta_{p1}\left(\mu_p^{(2)}/\mu_p^{(1)} - 1\right)} \tag{132}$$

They lead to the following estimate of the thin plate's rigidity

$$\begin{aligned} & D_2 + \frac{c_1\left(\kappa_p^{(1)} - \kappa_p^{(2)}\right)}{1 + c_2\alpha_{p2}\left(\kappa_p^{(1)}/\kappa_p^{(2)} - 1\right)} + \frac{c_1\left(\mu_p^{(1)} - \mu_p^{(2)}\right)}{1 + c_2\beta_{p2}\left(\mu_p^{(1)}/\mu_p^{(2)} - 1\right)} \\ & \leq D_p \leq \\ & D_1 + \frac{c_2\left(\kappa_p^{(2)} - \kappa_p^{(1)}\right)}{1 + c_1\alpha_{p1}\left(\kappa_p^{(2)}/\kappa_p^{(1)} - 1\right)} + \frac{c_2\left(\mu_p^{(2)} - \mu_p^{(1)}\right)}{1 + c_1\beta_{p1}\left(\mu_p^{(2)}/\mu_p^{(1)} - 1\right)} \end{aligned} \tag{133}$$

At least in mathematical formality, Hashin–Shtrikman bounds in linear elasticity provide estimates for the plate's rigidity based on "folk wisdom", if the applicability is granted, though these estimates are not optimal in general. The interesting part is that, there exists a direct comparison between the congruous bounds and the Hashin–Shtrikman bounds. For the case of two phase composite plates, this could be done by taking the thick plate as a 3D isotropic elastic medium, and first evaluating the bulk and shear moduli via Hashin–Shtrikman bounds

$$\lambda_{\text{eff}} = \lambda_{\text{eff}}(\lambda_1, \lambda_2, c_1, c_2) \tag{134}$$

$$G_{\text{eff}} = G_{\text{eff}}(G_1, G_2, c_1, c_2) \tag{135}$$

and then deriving the effective Young's modulus and Poisson's ratio as

$$E_{\text{eff}} = \frac{G_{\text{eff}}(3\lambda_{\text{eff}} + 2G_{\text{eff}})}{\lambda_{\text{eff}} + G_{\text{eff}}} \tag{136}$$

$$v_{\text{eff}} = \frac{\lambda_{\text{eff}}}{2(\lambda_{\text{eff}} + G_{\text{eff}})} \tag{137}$$

and finally the effective flexural rigidity,

$$D_{\text{eff}} = \frac{E_{\text{eff}} h^3}{12\left(1 - v_{\text{eff}}^2\right)}. \tag{138}$$

The comparison between the congrnous bounds and the Hashin–Shtrikman bounds are displayed in Fig. 4(a) and (b). In the first examples, the Young's modulus of the E-glass fibers is: 70 GPa, and its Poisson's ratio is: 0.25; the Young's modulus of epoxy matrix is: 2.8 GPa and its Poisson's ratio is: 0.35. In the second example, the Young's modulus of the Graphite fibers is 230 GPa, and its Poisson's ratio is 0.26; whereas, the Young's modulus of the epoxy matrix is 3.19 GPa and its Poisson's ratio is 0.35. One may observe that first the congruous bounds and Hashin–Shtrikman bounds are very close in both examples; second, in both examples, the congruous bounds are tighter than Hashin–Shtrikman bounds, because the congruous bound gives higher estimate in lower bound, and it offers almost the same estimation on upper bound (see: Fig. 4(a) and (b)). This makes sense because based on the Love–Kirchhoff assumption, the thin plate should be stiffer than the unconstrained 3D elastic continuum.

4.2. Self-consistent estimate

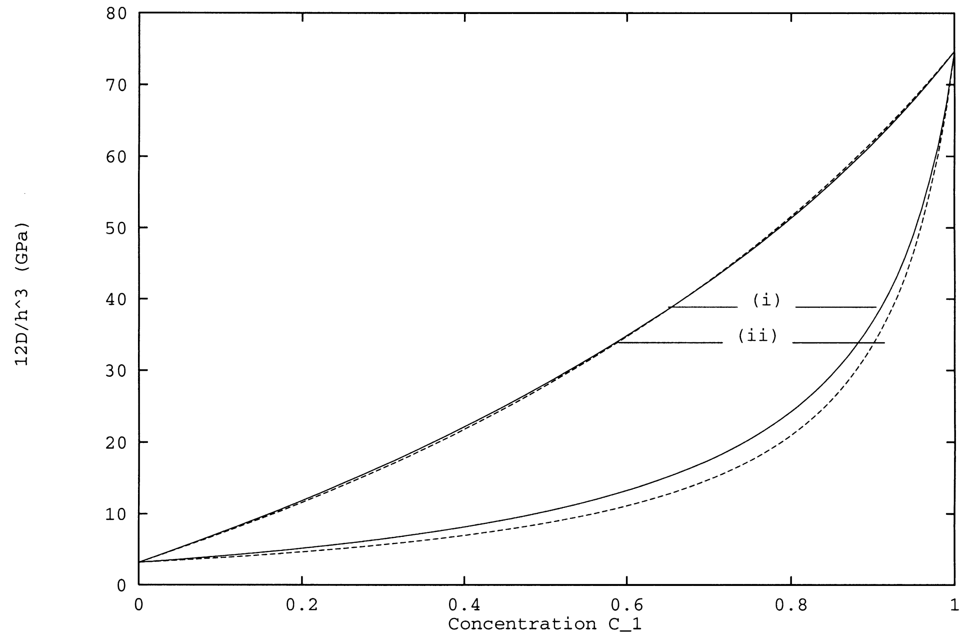
Because of the similarity between the mathematical structure of thin plate's governing equations and the governing equations of linear elasticity, the general framework of self-consistent approximation within the realm of linear elasticity should be valid in the thin plate theory as well. One can simply replace "eigenstrain" to "eigen-curvature" and "transformation stress" into "transformation moment". Of course, the physical interpretation on elastic stiffness is different, and most of all, the transformation tensor $\mathbf{P}$ is entirely different in quantity. Based on the equivalent inclusion theory (Hill, 1965; Budiansky, 1965), we assume that there exists a overall constraint stiffness, $\mathbf{L}^*$, or compliance, $\mathbf{N}^*$, such that in each phase of the plate

$$\bar{\mathbf{m}}_r - \bar{\mathbf{m}} = \mathbf{L}^*\left(\bar{\mathbf{k}} - \bar{\mathbf{k}}_r\right) \tag{139}$$

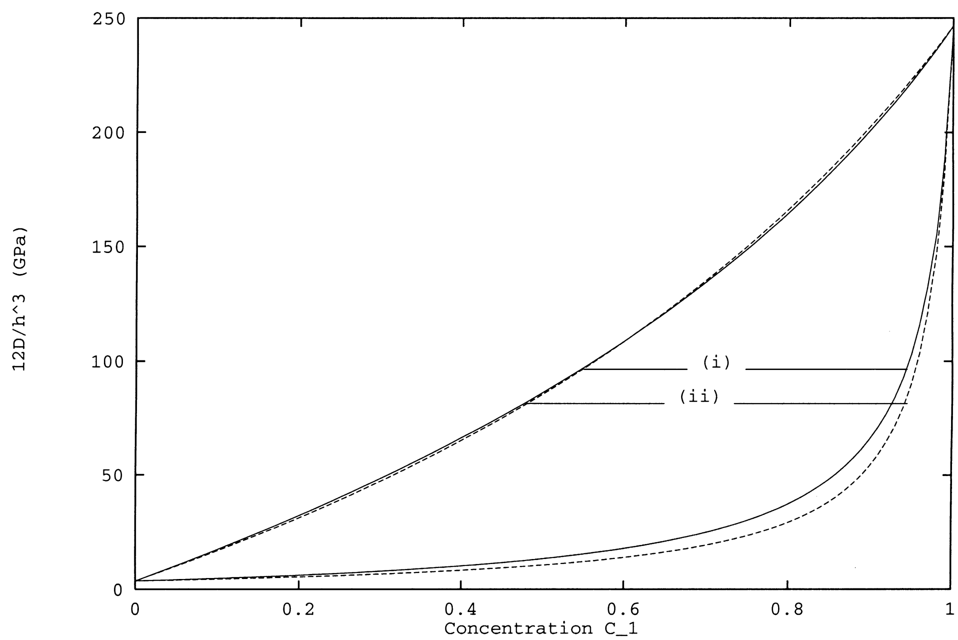
$$\bar{\mathbf{k}}_r - \bar{\mathbf{k}} = \mathbf{N}^*(\bar{\mathbf{m}} - \bar{\mathbf{m}}_r) \tag{140}$$

or

$$(\mathbf{L}^* + \mathbf{L}_r)\bar{\mathbf{k}}_r = (\mathbf{L}^* + \mathbf{L})\bar{\mathbf{k}} \tag{141}$$



(a) E-glass with epoxy matrix



(b) Graphite fibers with epoxy matrix

Fig. 4. Comparison between the congruous bounds and Hashin–Shtrikman bounds on the thin plate's rigidity: (a) the solid line: congruous bound; (b) the dash line: Hashin–Shtrikman bound.

$$(\mathbf{N}^* + \mathbf{N}_r)\bar{\mathbf{m}}_r = (\mathbf{N}^* + \mathbf{N})\bar{\mathbf{m}}. \tag{142}$$

Here, the physical quantities are the average moments and average curvatures. Apparently, this postulate makes sense in the generalized eigenstrain formulation. Furthermore, from Eqs. (141) and (142), one may derive,

$$\sum_{r=1}^{n} \frac{c_r}{\mathbf{L}_r + \mathbf{L}} = \frac{1}{\mathbf{L}^* + \mathbf{L}} = \mathbf{P} \Longrightarrow \sum_{r=1}^{n} c_r \left[(\mathbf{L}_r - \mathbf{L})^{-1} + \mathbf{P}\right]^{-1} = 0 \tag{143}$$

$$\sum_{r=1}^{n} \frac{c_r}{\mathbf{N}_r + \mathbf{N}} = \frac{1}{\mathbf{N}^* + \mathbf{N}} = \mathbf{Q} \Longrightarrow \sum_{r=1}^{n} c_r \left[(\mathbf{N}_r - \mathbf{N})^{-1} + \mathbf{Q}\right]^{-1} = 0 \tag{144}$$

or in the equivalent forms

$$\mathbf{L} = \left\{ \sum_{r=1}^{n} c_r \mathbf{L}_r \left[\mathbf{I} + \mathbf{P}(\mathbf{L}_r - \mathbf{L})\right]^{-1} \right\} \left\{ \sum_{s=1}^{n} c_s \left[\mathbf{I} + \mathbf{P}(\mathbf{L}_s - \mathbf{L})\right]^{-1} \right\}^{-1} \tag{145}$$

$$\mathbf{N} = \left\{ \sum_{r=1}^{n} c_r \mathbf{N}_r \left[\mathbf{I} + \mathbf{Q}(\mathbf{N}_r - \mathbf{N})\right]^{-1} \right\} \left\{ \sum_{s=1}^{n} c_s \left[\mathbf{I} + \mathbf{Q}(\mathbf{N}_s - \mathbf{N})\right]^{-1} \right\}^{-1} \tag{146}$$

To entertain the thought that this thin plate version's self-consistent scheme is practically useful, in what follows, we calculate the elastic stiffness for a special two-phase composite plate — a sheet of "Swiss cheese" (Fig. 5), in other words, one phase of the composite (say phase 1) is taken as cavity, which implies that $\mathbf{L}^{(1)} = 0$, or $\kappa_p^{(1)} = 0 = \mu_p^{(1)} = 0$. For two phase composite plate, Eq. (143) takes the form

$$\frac{c_1}{\mathbf{L} - \mathbf{L}_2} + \frac{c_2}{\mathbf{L} - \mathbf{L}_1} = \mathbf{P} \tag{147}$$

where

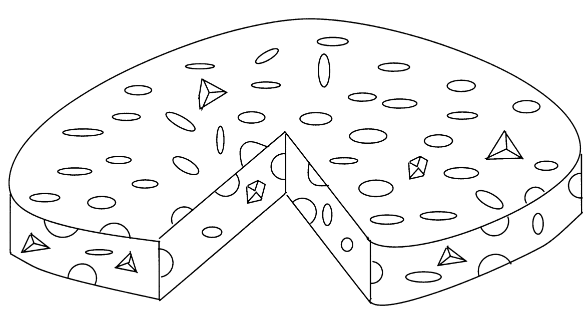


Fig. 5. A piece of "Swiss cheese", an example of thin plate with distributed cavities.

$$\mathbf{P} = \left(\frac{1}{2\kappa_p + 2\mu_p}, \frac{1}{4\kappa_p + 4\mu_p} \right) = \left(\frac{\alpha_p}{2\kappa_p}, \frac{\beta_p}{2\mu_p} \right) \tag{148}$$

Let $\kappa_p^{(1)} = \mu_p^{(1)} = 0$ We can solve κ_p in terms of μ_p i.e.

$$\kappa_p = \frac{c_2 \kappa_p^{(2)} \mu_p}{c_1 \kappa_p^{(2)} + \mu_p} \tag{149}$$

and μ_p satisfies the following quadratic equation,

$$\mu_p^2 + \left[(1 + c_2)\kappa_p^{(2)} + (1 - 2c_2)\mu_p^{(2)} \right] \mu_p + (1 - 3c_2)\kappa_p^{(2)} \mu_p^{(2)} = 0 \tag{150}$$

If the matrix is incompressible, $\kappa_p^{(2)} \to \infty$, we have

$$\mu_p = \frac{(3c_2 - 1)}{1 + c_2} \mu_p^{(2)} \tag{151}$$

$$\kappa_p = \frac{c_2}{c_1} \frac{(3c_2 - 1)}{(1 + c_2)} \mu_p^{(2)} \tag{152}$$

and

$$D_p = \frac{(3c_2 - 1)}{(1 + c_2)c_1} \mu_p^{(2)} \tag{153}$$

Obviously, the volume of the cavity should not exceed two third of the total volume of the plate. It is interesting to note that the similar problem solved in the context of linear elasticity (see Willis, 1981), the total cavity volume should not excess to one half of the total volume of the composite medium.

5. Conclusions

The basic idea of the approach taken in this study is that, one assumes the establishment of thin plate theory first, or a priori, then proceeds homogenization posteriorly; whereas, the conventional thought implicitly assumes that homogenization should be performed first, whereas, the establishment of structure theories comes second. By reversing the order, this paper presents a synergetic approach of a micromechanics that is congruous with the Love–Kirchhoff plate theory. It is demonstrated that there is parallel structure between the micromechanics in linear elasticity and the Love–Kirchhoff plate theory. This type analogue, we believe, exists in other ad hoc structural theories as well, such as Reissner–Mindlin plate theory (see Li, 1999) and linear elastic shell theory, though the mathematical structure might be more complex than what have been shown here.

To study and to develop micromechanics models that are congruous with the particular structural mechanics will provide engineers alternative methods to analyze associated composite structural components. It is fair to say that this new approach may shed some new light on understanding composite structures through structural mechanics point of view, rather than just from the viewpoint of material science. This contribution provides, in the first time, a congruous estimate of elastic stiffness of elastic thin plate. Even though the formulas derived in this paper resemble the classical formulas of linear elasticity in character, they obviously differ in quantities. Nevertheless, the application of this theory is subjected to certain restrictions, such as distribution patterns of inhomogeneities.

Acknowledgements

The author would like to thank Professor Toshio Mura, who introduces the author to the research field of micromechanics, and, in particular, this specific topic.

Appendix A. Evaluation of tensor $S_{\alpha\beta\zeta\eta}$

Based on formula (35), we calculate the tensor $S_{\alpha\beta\zeta\eta}$.

$$S_{1111} = \frac{(3-\nu)}{4\pi} I_1 - \frac{(1-\nu)a_1^2}{2\pi} I_{11} \tag{154}$$

$$S_{1122} = \frac{(1+\nu)}{4\pi} I_1 - \frac{(1-\nu)a_2^2}{2\pi} I_{12} \tag{155}$$

$$S_{1212} = S_{2121} = \frac{(1-\nu)}{8\pi}(I_1 + I_2) - \frac{(1-\nu)}{4\pi}\left(a_1^2 + a_2^2\right) I_{12} \tag{156}$$

$$S_{2211} = \frac{(1+\nu)}{4\pi} I_2 - \frac{(1-\nu)a_1^2}{2\pi} I_{12} \tag{157}$$

$$S_{2222} = \frac{(3-\nu)}{4\pi} I_2 - \frac{(1-\nu)a_2^2}{2\pi} I_{22} \tag{158}$$

$$S_{1112} = S_{1121} = 0 \tag{159}$$

$$S_{2212} = S_{2221} = 0 \tag{160}$$

$$S_{1211} = S_{2111} = 0 \tag{161}$$

$$S_{1222} = S_{2122} = 0 \tag{162}$$

where

$$I_\mu = \frac{1}{a_\mu^2} \int_0^{2\pi} \frac{l_\mu^2}{g}\, d\theta, \quad \mu = 1, 2 \tag{163}$$

$$I_{\alpha\beta} = \frac{1}{a_\alpha^2 a_\beta^2} \int_0^{2\pi} \frac{l_\alpha^2 l_\beta^2}{g}\, d\theta, \; \alpha, \beta = 1, 2 \tag{164}$$

which satisfy the following identities

$$I_1 + I_2 = 2\pi \tag{165}$$

$$a_1^2 I_{11} + \left(a_1^2 + a_2^2\right) I_{12} + a_2^2 I_{22} = 2\pi \tag{166}$$

$$\frac{a_1^2}{a_2^2} I_{11} + 2I_{12} + \frac{a_2^2}{a_1^2} I_{22} = \frac{1}{a_2^2} I_1 + \frac{1}{a_1^2} I_2 \tag{167}$$

In Qin et al. (1991), both expressions of S_{1111} and S_{2222} involve with I_{12}, which seems to be unreasonable.

Appendix B. Derivation and evaluation of integral operator $\sum$

Consider the fundamental solution of the governing equation:

$$L^0_{\alpha\beta\zeta\eta} G^{\infty}_{,\alpha\beta\zeta\eta}\left(\mathbf{x}, \mathbf{x}'\right) + \delta(\mathbf{x} - \mathbf{x}') = 0 \tag{168}$$

or $M^{G^{\infty}}_{\alpha\beta,\ \alpha\beta}(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$. The corresponding Green's function is

$$G^{\infty}\left(\mathbf{x}, \mathbf{x}'\right) = -\frac{1}{8\pi D^0} r^2 \ln r \tag{169}$$

Consider the subsidiary equation

$$\left(L^0_{\alpha\beta\zeta\eta} k^1_{\zeta\eta} + \tau_{\alpha\beta}\right)_{,\alpha\beta} = 0 \tag{170}$$

with the boundary conditions $w^1(\mathbf{x}) = 0$, $\frac{\partial w^1}{\partial n}(\mathbf{x}) = 0$, $\forall \mathbf{x} \in \partial\Omega$. Multiplying subsidiary equation (170) with the Green function (169) and integrating it by using Gauss theorem, one will have

$$w^1(\mathbf{x}) = -\int\int_{\Omega} G^{\infty}_{,\alpha\beta}(\mathbf{x}' - \mathbf{x})\left(\tau_{\alpha\beta} - \langle\tau_{\alpha\beta}\rangle\right)(\mathbf{x}')\, \mathrm{d}\Omega' + \oint_{\partial\Omega}\left\{-L^0_{\alpha\beta\zeta\eta} w^1_{,\zeta\eta} + \tau_{\alpha\beta} - \langle\tau_{\alpha\beta}\rangle\right\} n_\alpha G^{\infty}_{,\beta}\, \mathrm{d}S$$
$$- \oint_{\partial\Omega}\left\{-L^0_{\alpha\beta\zeta\eta} w^1_{,\zeta\eta\alpha} + \tau_{\alpha\beta,\ \alpha}\right\} n_\beta G^{\infty}\, \mathrm{d}S \tag{171}$$

From Eq. (170), we know that

$$\oint_{\partial\Omega}\left\{-L^0_{\alpha\beta\zeta\eta} w^1_{,\zeta\eta\alpha} + \tau_{\alpha\beta,\alpha}\right\} n_\beta\, \mathrm{d}S = 0 \tag{172}$$

Then, it is plausible that $T = \{-L^0_{\alpha\beta\zeta\eta} w^1_{,\zeta\eta\alpha} + \tau_{\alpha\beta,\ \alpha}\} n_\beta$ oscillates about zero. The same will be true for the term $\tau_{\alpha\beta} - \langle\tau_{\alpha\beta}\rangle) n_\beta$. It will be most likely that the term $L^0_{\alpha\beta\zeta\eta} w^1_{,\zeta\eta\alpha} n_\beta$ too oscillates around zero, because both, $\partial^2 w^1/\partial s \partial n$ and $\partial^2 w^1/\partial s^2$ are actually zero on the boundary. Thus, by the Saint–Venant principle[2], in the interior region

[2] A plate version of Saint–Venant principle is presented in Berdichevskii (1973).

$$w^1(\mathbf{x}) = -\iint_{\Omega} G^{\infty}_{,\alpha\beta}(\mathbf{x}',\mathbf{x})\big(\tau_{\alpha\beta} - \langle\tau_{\alpha\beta}\rangle\big)(\mathbf{x}')\,\mathrm{d}\Omega' \tag{173}$$

and subsequently,

$$k^1_{\alpha\beta} := -\sum \boldsymbol{\tau} = -\iint_{\Omega} k^{G^{\infty}}_{\alpha\beta,\ \lambda\mu}(\mathbf{x}' - \mathbf{x})\big(\tau_{\lambda\mu} - \langle\tau_{\lambda\mu}\rangle\big)(\mathbf{x}')\,\mathrm{d}\Omega' \tag{174}$$

Remark 7.1. *Unlike in three-dimensional elasticity, where there is only one boundary integral, there are two boundary integrals in* (171). *The argument made for neglecting the first boundary integral is probably not convincing. On the other hand, however, consider the fact*

$$\begin{aligned} w_{,\lambda\mu} = {} & -\iint_{\Omega} G^{\infty}_{,\alpha\beta\lambda\mu}\big(\tau_{\alpha\beta} - \langle\tau_{\alpha\beta}\rangle\big)\,\mathrm{d}\Omega + \oint_{\partial\Omega}\Big\{-L^0_{\alpha\beta\zeta\eta}w^1_{,\zeta\eta} + \tau_{\alpha\beta} - \langle\tau_{\alpha\beta}\rangle\Big\}n_{\alpha}G^{\infty}_{,\beta\lambda\mu}\,\mathrm{d}S \\ & -\oint_{\partial\Omega}\Big\{-L^0_{\alpha\beta\zeta\eta}w^1_{,\zeta\eta\alpha} + \tau_{\alpha\beta,\ \alpha}\Big\}n_{\beta}G^{\infty}_{,\lambda\mu}\,\mathrm{d}S \end{aligned} \tag{175}$$

Assume the boundary $\partial\Omega$ is in the remote region, $G^{\infty}_{,\beta\lambda\mu}\to 0$, the contribution from the first boundary integral is insignificant anyway.

Remark 7.2. *In both* Willis (1977, 1981), *the author argued that the effects of those boundary integrals are restricted only in a boundary layer around the boundary of the continuum, such that, for interior points of the domain, those effects can be neglected because they are away from the boundary. Nevertheless, if one views a thin plate as a 3D continuum as part of the boundary, both its upper and lower surface are very close to each other, no interior point within the plate is truly far away from the boundary. Thus, characterized as a lower order manifold, the thin plate presents itself as an exception to those arguments. It, therefore, seems to us that this is another incentive to develop a thin plate version micromechanics theory.*

Next, assume that the distribution of inhomogeneities is macroscopically isotropic and statistically homogeneous. Therefore, we can evaluate $\sum\boldsymbol{\tau}$ inside a circular region Ω_{r_a}, in which $\tau_{\alpha\beta} - \langle\tau_{\alpha\beta}\rangle$ is considered to be constant. Thus, one only need to evaluate

$$\boldsymbol{P}_0 := \iint_{\Omega_{r_a}} \Sigma^{\infty}(\mathbf{x})\,\mathrm{d}\Omega = \iint_{\Omega_{r_a}} k^{G^{\infty}}_{\alpha\beta,\ \zeta\eta}\,\mathrm{d}\Omega \tag{176}$$

where Ω_{r_a} is a circular region. By utilizing the result in Section 2, after some calculations, it is not difficult to find that

$$\begin{aligned} P_{\alpha\beta\zeta\eta} &= \iint_{\Omega_{r_a}} k^{G^{\infty}}_{\alpha\beta,\ \zeta\eta}(\mathbf{x})\,\mathrm{d}\Omega = \frac{1}{8(\kappa^0_p + \mu^0_p}\big\{\delta_{\alpha\eta}\delta_{\beta\zeta} + \delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\beta}\delta_{\zeta\eta}\big\} \\ &= \left[\frac{1}{2\left(\kappa^0_p + \mu^0_p\right)}\mathbf{J}_{\alpha\beta\zeta\eta} + \frac{1}{4\left(\kappa^0_p + \mu^0_p\right)}\mathbf{K}_{\alpha\beta\zeta\eta}\right] \end{aligned} \tag{177}$$

that is

$$\mathbf{P}_0 = \frac{1}{2\kappa_p^0 + 2\kappa_p^*}\mathbf{J} + \frac{1}{2\mu_p^0 + 2\mu_p^*}\mathbf{K} \tag{178}$$

with $\kappa_p^* := \mu_p^*$, $\mu_p^* := 2\kappa_p^0 + \mu_p^0$.

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